Homework #10 PHY523 Fall 2012 Due 12/6/2012

Reading Assignment: Please read Chapter 13

Problem #1
Pre-Quantum

In this question we "manipulate" the harmonic oscillator Hamiltonian. This is an illustration of the manner in which classical mechanics was first formally introduced. Some of these things should look familiar to you, if you have studied the quantum harmonic oscillator!

We consider a Hamiltonian of a 1D harmonic oscillator:

$$H = \frac{1}{2m}\left[p^2 + m^2\omega^2 q^2\right] \tag{1}$$

(a) Express H as a function of

$$a = \sqrt{\frac{m\omega}{2}}\left(q + i\frac{p}{m\omega}\right), \qquad a^* = \sqrt{\frac{m\omega}{2}}\left(q - i\frac{p}{m\omega}\right). \tag{2}$$

Make sure you simplify your answer.

(b) There is a mathematical operation referred to as "Poisson Brackets." For two functions A(p,q), and B(p,q), we define the poisson bracket as

$$[A, B] \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} \tag{3}$$

Calculate the Poisson brackets for: $[a, a^*]$, $[H, a]$, and $[H, a^*]$. You may find the following relationship useful: $[AB, C] = A[B, C] + B[A, C]$ for any functions A, B, and C.

(c) Show that for an arbitrary function $f(q, p, t)$, the following relationship is true:

$$\dot{f} = \frac{\partial f}{\partial t} + [f, H] \tag{4}$$

(d) Write down the equations of motion for $a(p, q)$ and $a^*(p, q)$ and solve them for arbitrary initial conditions.

Problem #2
Phase-Space

Find the hamiltonian, $\mathcal{H}$ for a mass m confined to the x axis and subject to a force $F = -kx^3$ where $k > 0$. Sketch and describe the phase-space orbits.

Problem #3
Liouville's Theorem

A beam of protons is moving along an accelerator pipe in the z-direction. The particles are uniformly distributed in a cylindrical volume of length L_0 (in the z direction) and radius R_0. The particles have momenta uniformly distributed with p_z in an interval $p_0 \pm p_z$ and the transverse (along x-y) momentum $p_\perp$ in a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing mechanism does not affect the length of the bunch of particles moving through the pipe, or the spread in p_z).