Homework #6 PHY523 Fall 2017 Due 10/10/2017

Problems marked with as asterisk are more challenging and/or calculationally involved. Do the other ones first, if you prefer. Come to me or talk to each other if you would like help/hints. I encourage discussion among you, but all submitted work must be your own.

Problem #1
Viral Theorem

(a) A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U = kr^n$. Prove the *Virial Theorem*, that $T = nU/2$.

(b) There is a more general version of this theorem that applies to any periodic orbit of a particle: Find the time derivative of the quantity $G = \mathbf{r} \cdot \mathbf{p}$, and, by integrating from time 0 to t, show that

$$\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \mathbf{F} \cdot \mathbf{r} \rangle \tag{1}$$

where $\mathbf{F}$ is the net force on the particle, and $\langle f \rangle$ denotes the average over time of any quantity f.

(c) Use this result to prove that if $\mathbf{F}$ arises from a potential energy $U = kr^n$, then $\langle T \rangle = n\langle U \rangle/2$, so long as we take the average over a very long time.

Problem #2
Conservation Laws

This question shows the power of conserved quantities. Consider a particle in a smooth monotonic attractive central potential $V(r)$. At time $t = 0$ the particle is at a point $\vec{r}_0$ and has velocity $\vec{v}_0$. It is given that the velocity is inward, that is, $r(t = \epsilon) < r(t = 0)$ (where, as always, it is implicit that ϵ is very small compared to all other time scales in the problem).

(a) The two conserved quantities in the problem ar E and $\vec{L}$. Calculate them based on the initial conditions.

(b) In terms of angles, the above conserved quantities should depend only on the angle between $\vec{r}_0$ and $\vec{v}_0$. Verify that this is indeed the case in your answer and explain why.

(c) We try to find the point the particle is closest to the origin, $r_{\min}$. What is the value of the radial velocity at that point, $\dot{r}|_{r_{\min}}$? What is the angle between $\vec{r}$ and $\vec{v}$ there?

(d) Using the value of the conserved quantities, write the equations that can be used to find $r_{\min}$.

Problem #3
A symmetric well potential

Consider a particle of mass m in the following one dimensional potential

$$V(x) = ax^{2n}, \tag{2}$$

where n is a positive integer and $a > 0$.

(a) What are the units of a?

(b) Calculate the period of the particle as a function of its energy. You will need the following integral

$$\int_{-1}^{1} dx \left(1 - x^{2n}\right)^{-1/2} = 2\sqrt{\pi}\, \Gamma\left(1 + \frac{1}{2n}\right) \Gamma\left(\frac{n+1}{2n}\right)^{-1}. \tag{3}$$

(c) Show that the period is independent on the energy only for the harmonic case, that is, $n = 1$.