Homework #2 PHY523 Fall 2017 Due 9/12/2017

Problems marked with as asterisk are more challenging and/or calculationally involved. Do the other ones first, if you prefer. Come to me or talk to each other if you would like help/hints. I encourage discussion among you, but all submitted work must be your own.

Problem #1
Fermat's Principle

Fermat's principle states that "the path taken between two points by a ray of light is the path that can be traversed in the least time." That is, light "chooses" the fastest path. Snell's law tells us how light changes direction when it moves between two different materials with different index of refraction. Snell's law reads

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}, \tag{1}$$

where θ_i is the angle with respect to the normal to the surface of the interface between the materials. Derive Snell's law from Fermat's principle.

Problem #2 *
Center of Mass

Prove that the magnitude R of the position vector for the center of mass of a system of N particles, each with mass m_α, and position vector r_α is given by:

$$M^2R^2 = M\sum_{\alpha=1}^{N} m_\alpha r_\alpha^2 - \frac{1}{2}\sum_{\alpha,\beta=1}^{N} m_\alpha m_\beta(\vec{r}_\alpha - \vec{r}_\beta)^2 \tag{2}$$

Problem #3
Chain Rule

Working with generalized coordinates is very useful. Yet, many times, expressions are more complicated in such cases. You always have to be careful to write the correct physical expression using the right variables. Here we work a bit with polar coordinates.

Spherical polar coordinates (r, θ, φ) are defined by

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \tag{3}$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

1. Show that

$$d\mathbf{r} \equiv dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\varphi\mathbf{e}_\varphi, \tag{4}$$

   where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit vectors. Find the explicit form of these vectors on the basis $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. Note that there are two ways to do it. One is to use the chain rule. That is

$$d\mathbf{r} = d(r\mathbf{e}_r) = dr\mathbf{e}_r + r\frac{\partial\mathbf{e}_r}{\partial r}dr + r\frac{\partial\mathbf{e}_r}{\partial\theta}d\theta + r\frac{\partial\mathbf{e}_r}{\partial\varphi}d\varphi \tag{5}$$

   the other way, is to directly differentiate $d\mathbf{r}$ in Cartesian coordinates. You can choose either way, or to do both.

2. Express the velocity, $\dot{\mathbf{r}}$, the kinetic energy, $m\dot{\mathbf{r}}^2/2$, and acceleration, $\ddot{\mathbf{r}}$, of a particle in spherical coordinates, by expressing the time derivatives $\dot{\mathbf{e}}_r$, $\dot{\mathbf{e}}_\theta$ and $\dot{\mathbf{e}}_\varphi$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$.

Remark: usually, the notation for a unit vector in the x direction is given by $\hat{x}$. Here, I used homework from someone else where a different notation is used, $\mathbf{e}_x$. Of course, this is just notation. It is a good time to mention the following quote from H. Georgi (Weak interactions, page 2): "Throughout, I will switch around often among notations without warning or apology. In the long run this will make your life easier. Notation should be your slave, not the other way around."

Problem #4
Asteroids!

A spherical asteroid of uniform density with initial radius R_0 is spinning with angular velocity ω_0. As the eons go by, pieces of cosmic detritus stick the asteroid (much like the example of the spitball and the turntable in lecture), and the asteroid, (while remaining spherical, and of uniform density) becomes bigger, with new radius R. Assuming that the additional matter was at rest relative to the asteroid, what is the new angular velocity of the asteroid? You should find useful the fact that the moment of inertia of a uniformly dense sphere (about any axis which passes through its center) is $I_{\text{sphere}} = \frac{2}{5}MR^2$. What is the final angular velocity if $R = 2R_0$?

Problem #5
Computation

Take your Week 1 mathematica code for 3 moving bodies and add a computation of the total energy (kinetic plus potential energy) of the system of 3 bodies. Plot the total energy, the kinetic energy, and the potential energy as a function of time. Comment on your results.