Lecture 14.1 - Physics 523

Non-linearity + Chaos The principal reason we understand so little about chaotic systems is that non-linearity is a prerequisite for chaos

We know how to solve very few non-linear differential eq's

Pendulum $\quad mL^2\ddot{\phi} = -mgL\underbrace{\sin\phi}_{\text{non-linear}}$ $\qquad$ Gravity $\quad m\ddot{\vec{r}} = -\frac{GMm}{r^2}\hat{r}$ $\qquad$ Aside: we could solve 2-body problem by making it linear

Recall A feature of chaotic systems is strong sensitivity to initial conditions "The Butterfly Effect"

While non-linearity is a pre-req., not sufficient (single pendulum exhibits regular periodic motion)

Driven Damped Pendulum $\quad mL^2\ddot{\phi} = -mgL\sin\phi - \underbrace{bL^2\dot{\phi}}_{\text{damping force}} + \overbrace{LF(t)}^{\text{driving force}}$ $\qquad$ (Non-lagrangian, non-Hamiltonian)

Recall the superposition principle no longer applies! $\rightarrow$ only works for linear equations $\rightarrow$ No simple eigenvalue / eigenvector analysis ☹

Today, let us take the DDP as our canonical example $\quad$ assume $F(t)$ is sinusoidal $\quad F(t) = F_0\cos\omega t$

Rewrite EoM $\quad \ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi = \frac{F_0}{mL}\cos\omega t \quad$ or $\quad \ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\sin\phi = \gamma\omega_0^2\cos\omega t$

$$\beta = \frac{b}{2m} \qquad \omega_0^2 = g/L \qquad \gamma = \frac{F_0}{mg}$$

Small initial $\phi, \dot{\phi}$, small γ $\rightarrow$ $\sin\phi \sim \phi$ + equation is nearly linear (and remains so, since small driving force would not generally pump much E in)

We studied this case already $\rightarrow$ steady state motion is $\phi = \underbrace{A\cos(\overbrace{\omega}^{\text{driving freq.}} t - \delta)}_{\text{expressible in terms of } \beta, \omega_0, \gamma}$ $\qquad$ Transients eventually damp away

Nearly linear $\sin\phi \sim \phi - \frac{\phi^3}{6}$ ← relatively small $\phi \sim A\cos(\omega t - \delta)$ Plug this in, see what we get $\phi^3 \propto \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$ ← "anomalous"

other terms ($\phi, \dot{\phi}, \ddot{\phi}$ must also have this dependence! (Since RHS does not)) $\phi(t) = A\cos\omega t + \lambda A^3 \cos 3(\omega t - \delta)$

A^3 suppressed for small oscillations

This new term is a "harmonic" (multiple of original frequency) Can keep going, including more and more (sequentially suppressed) harmonics

Long term motion is again periodic, but not exactly sinusoidal

Perturbation Theory → Requires small parameter (γ → driving force)

Increasing γ $\gamma \sim 1$ is the "tipping point" In text, Taylor steps through $\gamma = 1.06$ (I), 1.073 (II), 1.077 (III)

I. Motion begins w/ severe transients but settles into motion that (to high numerical accuracy) matches periodicity of driving force

II. Initial motion again wild, and settles into periodic motion, but period does not match driving force $T_{obs} = 2T_{driving}$

(but almost periodic w/ same period)

III. Long term motion is periodic, but behavior depends on initial conditions!
1) Motion w/ $T = 3T_d$ is dominant $\phi_0 = \dot{\phi}_0 = 0$
2) Motion w/ $T = 2T_d$ (T_d dominant) $\phi_0 = -\pi/2,\ \dot{\phi}_0 = 0$

Steady state behavior referred to as "attractor" solution → Some systems have multiple attractors!

$\boxed{\text{Period doubling}}$ As γ is increased, period of oscillation goes up by factors of 2 "Bifurcation"

→ Universal to large # of systems

When does period doubling occur? Experimentally, values of γ_{2n} come closer together

$$(\gamma_{n+1} - \gamma_n) \simeq \frac{1}{\delta}(\gamma_n - \gamma_{n-1}) \quad w/ \quad \delta = 4.669206 \quad \text{"Feigenbaum \#"}$$

Same # in many, many systems!

Note that this series converges rapidly ⇒ there is some γ_c at which $T_{steady} \to \infty$ $\gamma_c = 1.0829$ for DDP

Period doubling signals an onset of chaotic behavior (but does not occur in all chaotic systems)

$\boxed{\text{Sensitivity to initial conditions}}$ In the linear case, transients decay exponentially, this remains true through the period doubling regime, but after $\gamma > \gamma_c$, this is lost $\Delta\phi$ grows exponentially

displacements of identical oscillators, different I.C.'s

$\Delta\phi = \phi_2 - \phi_1$ → Total loss of long-term predictivity! 10^{-4} radian discrepancy grow to 2π disc. in just 16 cycles for $\gamma = 1.105$

Exponent of growth/decay of initial separation = Liapunov exponent $|\Delta\phi(t)| \sim Ke^{\lambda t}$ L. exp.

For even larger γ, system sometimes returns to stable behavior, but then returns quickly to chaos again