Lecture 14.1 - Physics 523

Chaos in Hamiltonian Systems

For a system w/ N d.o.f., can have function $\mathcal{H}(\vec{q},\vec{p},t) \to \dot{\vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}$ $\quad \dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}$

for $\mathcal{H} = \mathcal{H}(\vec{p},\vec{q}) \to \dot{\mathcal{H}} = 0$ (identify $\mathcal{H}$ w/ conserved total energy).

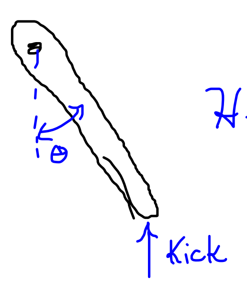
In terms of phase space vectors $\vec{z} = (\vec{q},\vec{p})$, $\quad \dot{\vec{z}} = \underbrace{\begin{pmatrix} 0_{N\times N} & -\mathbb{1}_{N\times N} \\ \mathbb{1}_{N\times N} & 0_{N\times N} \end{pmatrix}}_{S_N} \cdot (\vec{\nabla}_{PS}\, \mathcal{H})$

Liouville's Theorem: A collection of neighboring initial conditions $\vec{z}_0$ which span a phase space volume maintain constant volume

(phase space is "incompressible")

This is true even if $\mathcal{H} \neq$ constant!

Example the kicked rotor $\Rightarrow$ rod attached at pivot w/ no friction, gets kick in same direction periodically



$\mathcal{H} = \frac{p_\theta^2}{2I} + K\cos\theta \sum_n \delta(t - n\tau)$

ang. mom   strength of kick   time between kicks   moment of inertia

In between kicks, $p_\theta = $ const., $\dot{\theta} = \frac{p_\theta}{I} \Rightarrow$ motion just after each kick is all that is needed

$$\boxed{\theta_{n+1} = \left(\theta_n + \frac{p_n^\theta}{I}\tau\right) \bmod 2\pi}$$

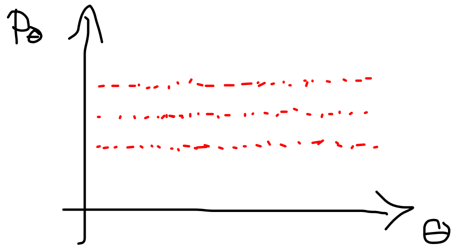
θ only def on $0 \leq \theta < 2\pi$

$\dot{p}_\theta = K \sum_n \sin\theta_n\, \delta(t - n\tau)$

$$\boxed{p_{n+1}^\theta = p_n^\theta + K\sin\theta_{n+1}}$$

Take $\tau = I \Rightarrow$ $\theta_{n+1} = (\theta_n + p_n) \bmod 2\pi$ $\quad p_{n+1} = p_n + K\sin\theta_{n+1}$ $\quad$ Tells us how to move point (or blob) in phase space forward in time via discrete time steps τ

Imagine small kicks → for $K=0$, rotor just spins w/ constant angular velocity. How does this look in phase space?



iterations cover lines of constant p_θ

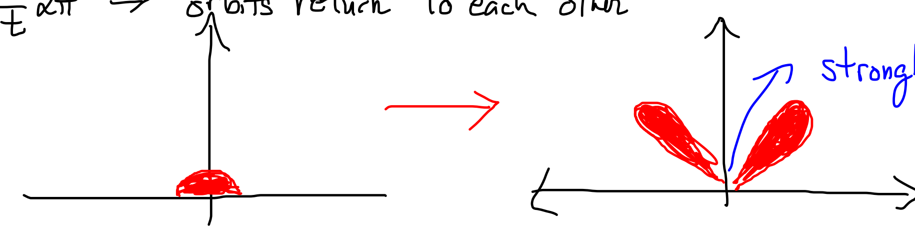
→ circle is covered

→ for $p_0 = \frac{s}{t} 2\pi$ (integers) → orbits return to each other

What happens when we take $K \neq 0$ but small?

→ for small momenta, small kicks are important

Consider a blob in phase space at $p_0=0$, $\theta=0$



strongly sensitive to initial conditions near origin

Now follow some large set of initial conditions for a large amount of time. for different values of K

→ for irrational q, w/ $p = q \cdot 2\pi$, orbits are stable under perturbation

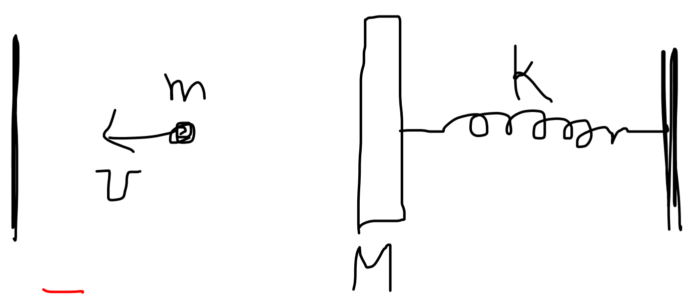
rational q, eg, $(\theta, p) = (\pi, 0), (\pi, 2\pi)$ and $(0,\pi) / (\pi,\pi)$

<u>Stat. mech and chaos</u> Why doesn't Jupiter kick the earth out of the solar system? Dof. are large, system is perturbed

On one hand, you have equipartion of energy, central tenet of stat mech.

on other have stable n-body systems

Another system



energy of this system is <u>constant</u>

Characterize system by $\omega_0 T$ ← avg time between bounces

characteristic freq. $\sqrt{\frac{K}{M}}$

For large $\omega_0 T$, get highly chaotic behavior

→ equipartion $\frac{1}{3}E$ in T_m, T_M, and U_{spring}