HW#9 Solutions

1) a) $\vec{\omega} = (\omega_0 \cos\Omega_b t, -\omega_0 \sin\Omega_b t, \omega_3)$ $\quad \vec{\omega}^2 = \omega_0^2(\cos^2 + \sin^2) + \omega_3^2 = \omega_0^2 + \omega_3^2 = \text{const.}$

b) in the space frame, $\Omega_s = \omega \dfrac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}$ where λ_i are principle moments. For the disk, $\lambda_3 = 2\lambda_{1,2}$

$$= \omega \frac{\sqrt{4\lambda_1^2 + (\lambda_1^2 - 4\lambda_1^2)\sin^2\alpha}}{\lambda_1} = \omega\sqrt{4 - 3\sin^2\alpha} \qquad \boxed{\Omega_s = \omega\sqrt{4 - 3\sin^2\alpha}}$$

2) a) For the double pendulum, the EoM are: $(m_1+m_2)L_1^2\ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 = -(m_1+m_2) g L_1 \phi_1$

$$m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2$$

in matrix form:

$$\underbrace{\begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}}_{\hat{M}} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -(m_1+m_2) g L_1 & 0 \\ 0 & -m_2 g L_2 \end{pmatrix}}_{-\hat{K}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Presume oscillatory solutions $\vec{\phi} = \vec{\phi}_0 e^{i\omega t}$

$(\hat{M}\omega^2 - \hat{K})\vec{\phi}_0 = 0 \quad$ I. $\det(\hat{M}\omega^2 - \hat{K}) = 0 \rightarrow$ used Mathematica $\quad \boxed{\omega^2 = \dfrac{g(L_1+L_2)(m_1+m_2)}{2m_1 L_1 L_2}\left[1 \pm \sqrt{1 - \dfrac{4 L_1 L_2 m_1}{(m_1+m_2)(L_1+L_2)^2}}\,\right]}$

b) for $m_1 = m_2$, $L_1 = L_2$ $\quad \omega^2 = 2\dfrac{g}{L}\left[1 \pm \dfrac{1}{\sqrt{2}}\right]$ $\quad$ which is in agreement w/ earlier result

c) As $m_2 \rightarrow 0$ $\quad \omega^2 = \dfrac{g(L_1+L_2)}{2L_1L_2}\left[1 \pm \sqrt{1 - \dfrac{4L_1L_2}{(L_1+L_2)^2}}\right] = \dfrac{g(L_1+L_2)}{2L_1L_2}\left[1 \pm \dfrac{|L_1 - L_2|}{L_1+L_2}\right] = \dfrac{g}{2L_1L_2}(L_1 + L_2 \pm |L_1 - L_2|) = \begin{cases} g/L_1 \\ g/L_2 \end{cases}$

For small m_2, first mass and 2nd mass decouple $\quad$ <u>mode 1</u> - first mass moves, 2nd is still

<u>mode 2</u> - 2nd mass moves, 1st mass still

3) displacement from equilibrium $d_1 = \sqrt{\ell^2 + y_1^2} - \ell \cong \dfrac{y_1^2}{2}$ $\quad d_2 = \dfrac{(y_1 - y_2)^2}{2}$ $\quad d_3 = \dfrac{y_2^2}{2}$

$$\mathcal{L} = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - \frac{T}{2}\left[y_1^2 + (y_1 - y_2)^2 + y_2^2\right]$$

EoM $\quad m\ddot{y}_1 = -T(y_1 - y_2)$ $\quad \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = -T\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\quad m\ddot{y}_2 = -T(y_2 - y_1)$

$\vec{y} = \vec{y}_0 e^{i\omega t} \Rightarrow \det\begin{pmatrix} m\omega^2 - 2T & T \\ T & m\omega^2 - 2T \end{pmatrix} = 0 = (m\omega^2 - T)(m\omega^2 - 3T)$ $\qquad \omega^2 = \begin{cases} T/m \\ 3T/m \end{cases}$

<u>Mode 1</u> $\quad \omega_1^2 = T/m \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} = 0 \Rightarrow$ mode is $\quad \vec{y}_1 = \dfrac{A_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(\omega_1 t - \delta_1)}$

<u>Mode 2</u> $\quad \omega_2^2 = \dfrac{3T}{m} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} = 0$ mode is $\quad \vec{y}_2 = \dfrac{A_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i(\omega_2 t - \delta_2)}$

$$\boxed{y(t) = \frac{A_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(\omega_1 t - \delta_1)} + \frac{A_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i(\omega_2 t - \delta_2)}}$$