$\boxed{\text{HW\#8 Solutions}}$

1) a) There are 3 degrees of freedom. Two to specify the position of the center of mass, one to specify orientation.

b) There is only one rotation angle, so only one component of angular momentum

c) This is just the result from 3D... $\ell = \vec{r}\times\vec{p} = mrv\sin\phi$

d) ω is $\dot{\phi}$, and $\vec{v}_{rel} = \omega r\hat{\phi}$, so we just use vector addition to see $\vec{v} = \vec{V}_{cm} + \omega r\hat{\phi}$

e) $T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(\vec{V}+r\omega\hat{\phi})^2 = \frac{1}{2}m(V_{cm}^2 + r^2\omega^2) \Rightarrow T_{tot} = \frac{1}{2}\sum_i m_i\left(V_{cm}^2 + (x_i^2+y_i^2)\omega^2\right) = \frac{1}{2}\underbrace{M}_{\sum_i m_i}V_{cm}^2 + \frac{1}{2}\underbrace{I}_{\sum_i m_i(x_i^2+y_i^2)}\omega^2$

for single component of rigid body

f) $\mathcal{L} = T - V \Rightarrow \frac{\partial\mathcal{L}}{\partial X_{cm}} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}_{cm}} \Rightarrow \boxed{M\ddot{X} = -\frac{\partial V}{\partial X}\ \left(\text{Similarly, } M\ddot{Y}_{cm} = -\frac{\partial V}{\partial Y_{cm}}\right)}$ then there is the EoM concerning the behavior of the coordinate ϕ: $\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \Rightarrow -\frac{\partial V}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\omega} = I\dot{\omega} \Rightarrow \boxed{I\dot{\omega} = -\frac{\partial V}{\partial\phi}}$

g) If $V=0$ then the above eom for ϕ is $I\dot{\omega}=0$ so $\omega = $ const. (in 3D, $\vec{L} \not\propto \vec{\omega}$ in general, so this is not the case for 3D)

h) $\dot{\omega}=0 \Rightarrow \ddot{\phi}=0 \Rightarrow \phi = \phi_0 + \omega t$ for $\phi(0)=\frac{\pi}{4}$, $\dot{\phi}(0)=a$, $\phi_0 = \pi/4$, $\omega = a \Rightarrow \boxed{\phi(t) = \pi/4 + at}$

2) a) $I = ma^2\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$ b) $\det(I - \lambda\mathbb{1}) = (10ma^2-\lambda)\left[(6ma^2-\lambda)^2 - 16\right] = (10ma^2-\lambda)^2(2ma^2-\lambda)$ $\lambda = 10ma^2, 2ma^2$ Principal moments

Principal axes are $\overbrace{(1,0,0), (0,\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}})}^{\lambda = 10ma^2}$, $\overbrace{(0, 1/\sqrt{2}, 1/\sqrt{2})}^{\lambda = 2ma^2}$

3) $I_{earth} = \frac{2}{5}MR^2 = \frac{2}{5}(6\times10^{24}\,kg)(6.4\times10^6\,m)^2 \cong 1\times10^{38}\,kg\cdot m^2$ and $\vec{\Gamma} = \dot{\vec{L}} = \lambda_3\omega\dot{\hat{e}}_3 = \lambda_3\omega(\vec{\Omega}\times\hat{e}_3) \Rightarrow |\vec{\Gamma}| = \lambda_3\omega\Omega\sin\theta$

$|\vec{\Gamma}| = (10^{38}\,kg\cdot m^2)(7\times10^{-5}\,rad/s)(8\times10^{-12}\,rad/s)\sin\left(\frac{23}{180}\cdot\pi\right) = \boxed{2\times10^{22}\ N\cdot m}$