HW#10 Solutions

1) a) $\omega a a^* = \omega \sqrt{\frac{m\omega}{2}}\left(q + \frac{ip}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(q - \frac{ip}{m\omega}\right) = \frac{m\omega^2}{2}\left(q^2 + \frac{p^2}{m^2\omega^2}\right) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \mathcal{H}$ $\boxed{\mathcal{H} = \omega a a^*}$

b) $[a, a^*] = \sqrt{\frac{m\omega}{2}} \cdot \sqrt{\frac{m\omega}{2}} \cdot \frac{(-i)}{m\omega} - \sqrt{\frac{m\omega}{2}}\,\frac{i}{m\omega}\sqrt{\frac{m\omega}{2}} = -i$

$[\mathcal{H}, a] = [\omega a a^*, a] = \omega a [a^*, a] = +i\omega a$

$[\mathcal{H}, a^*] = [\omega a a^*, a^*] = \omega a^* [a, a^*] = -i\omega a^*$

c) $\frac{d f(p,q,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial f}{\partial t} + [f, \mathcal{H}]$

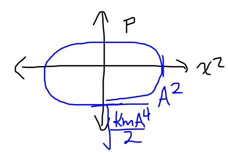
d) $\frac{da}{dt} = [a, \mathcal{H}] = -i\omega a \Rightarrow a(t) = a_0 e^{-i\omega t}$

$\frac{da^*}{dt} = [a^*, \mathcal{H}] = +i\omega a^* \Rightarrow a^*(t) = a_0^* e^{+i\omega t}$

2) $F = -kx^3 = -\frac{\partial U}{\partial x} \Rightarrow U = \frac{k}{4}x^4 \quad \left(T = \frac{p^2}{2m}, \text{ as usual}\right) \Rightarrow \mathcal{H} = \frac{p^2}{2m} + \frac{k}{4}x^4$ EoM $\dot{x} = \frac{p}{m}$ $\dot{p} = -kx^3$

$\mathcal{H}$ is bounded to be + def. + $\mathcal{H}$ = const. = E_{tot} + motion is periodic $E_{tot} = \frac{k}{4}A^4$ ← amplitude of oscillation

$\mathcal{H} = \frac{p^2}{2m} + \frac{k}{4}x^4 = \frac{k}{4}A^4 \Rightarrow \left(\frac{x}{A}\right)^4 + \frac{2p^2}{km A^4} = 1$ (ellipse in (p, x^2) plane



compressed ellipse (Jacobi elliptic) in (p, x) plane.

3) $V_{ps} = V_x \cdot V_p = (\pi R_0^2 L_0)(2\Delta p_z\, \pi (\Delta p_\perp)^2)$ + V_{ps} is conserved

squeezing R_0 comes at price of increasing $\Delta p_\perp$ (later times, large $\Delta p_\perp$ blows beam apart)

↖at same rate → $R = \frac{R_0}{2} \Rightarrow \Delta p_\perp^{new} = 2\Delta p_\perp$