

Lecture 13.2 - Physics 523

Hamiltonian Mechanics

Review

For any mechanical system we can define a function $\mathcal{H}(q_1, \dots, q_n; p_1, \dots, p_n)$ where the equations of motion follow from $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

★ For most systems we will deal with, \mathcal{H} is just total energy $\mathcal{H} = T(\vec{q}, \vec{p}) + U(\vec{q})$

This approach to mechanics is equivalent to both Newtonian + Lagrangian formulations

Time derivative of \mathcal{H}

Consider taking the total derivative of \mathcal{H} wrt. time: $\frac{d\mathcal{H}}{dt} = \sum_i \left[\underbrace{\frac{\partial \mathcal{H}}{\partial p_i}}_{+\dot{q}_i} \dot{p}_i + \underbrace{\frac{\partial \mathcal{H}}{\partial q_i}}_{-\dot{p}_i} \dot{q}_i \right] + \underbrace{\frac{\partial \mathcal{H}}{\partial t}}_{\text{explicit } t \text{ dependence}}$
 $= \frac{\partial \mathcal{H}}{\partial t}$

Hamilton's equations tell us \mathcal{H} can only depend on time explicitly \rightarrow not dep. through coordinates

★ If $\left. \begin{array}{l} 1) \text{ coordinates are natural } \vec{q}(\vec{r}) \text{ (not } \vec{q}(\vec{r}, t)) \\ 2) \mathcal{H} \text{ has no explicit time dependence} \end{array} \right\} \text{ then } \left. \begin{array}{l} 1) \mathcal{H} = T + U = \text{total energy} \\ 2) \text{ total } E \text{ is conserved} \end{array} \right\}$

Example I

Mass moving in 1D

$$T = \frac{p^2}{2m}$$

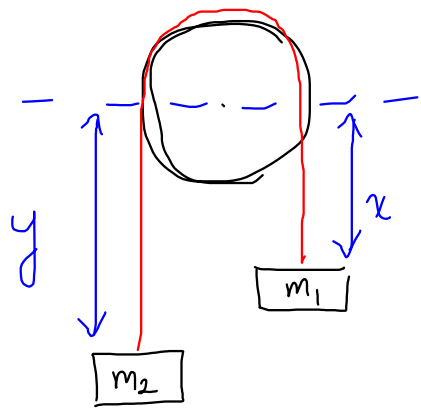
$$U = U(x)$$

$$\mathcal{H} = T + U = \frac{p^2}{2m} + U(x)$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial U}{\partial x}$$

Can now take $p = m\dot{x} \Rightarrow \dot{p} = m\ddot{x}$ and thus $m\ddot{x} = -\frac{\partial U}{\partial x} = F_x$ (as we already knew!)

Example II Atwood's Machine



1) Calculate $\mathcal{L}(x, \dot{x})$

2) calculate p

3) Calculate $\mathcal{H}(p, x)$

with $y = L - x$ 4) obtain Hamilton's eq's

$$1) \quad T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \quad U = -m_1gx - m_2gy$$

$$= -g(m_1 - m_2)x - m_2gL$$

$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + g(m_1 - m_2)x$$

$$2) \quad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x} \Rightarrow T = \frac{p^2}{2(m_1 + m_2)}$$

$$3) \quad \mathcal{H} = \frac{p^2}{2(m_1 + m_2)} - g(m_1 - m_2)x$$

$$4) \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{(m_1 + m_2)} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = +g(m_1 - m_2)$$

$$\ddot{x} = +g \frac{(m_1 - m_2)}{(m_1 + m_2)}$$

Particle in Central Potential

$$U = U(r) \quad T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

Note ϕ is cyclic w/ $p_\phi = mr^2\dot{\phi} = l$ (ang. mom.)

$$p_r = m\dot{r}$$

Immediately eliminate ϕ + $p_\phi = l$
(1D problem!)

$$\mathcal{H} = T + U = \frac{1}{2}m\left[\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{l}{mr^2}\right)^2\right] + U(r)$$

Hamilton's eq's:

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \quad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\frac{\partial U}{\partial r} + \frac{l^2}{mr^3}$$

Summary of procedure

1) pick coordinates q_1, \dots, q_n

2) get $T + U$ in terms of q, \dot{q}

3) get p 's ($\frac{\partial T}{\partial \dot{q}}$ for conservative potentials)

4) compute $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$ (or $T + U$ for natural coord.)

5) Write down Hamilton's eq.'s

Phase Space

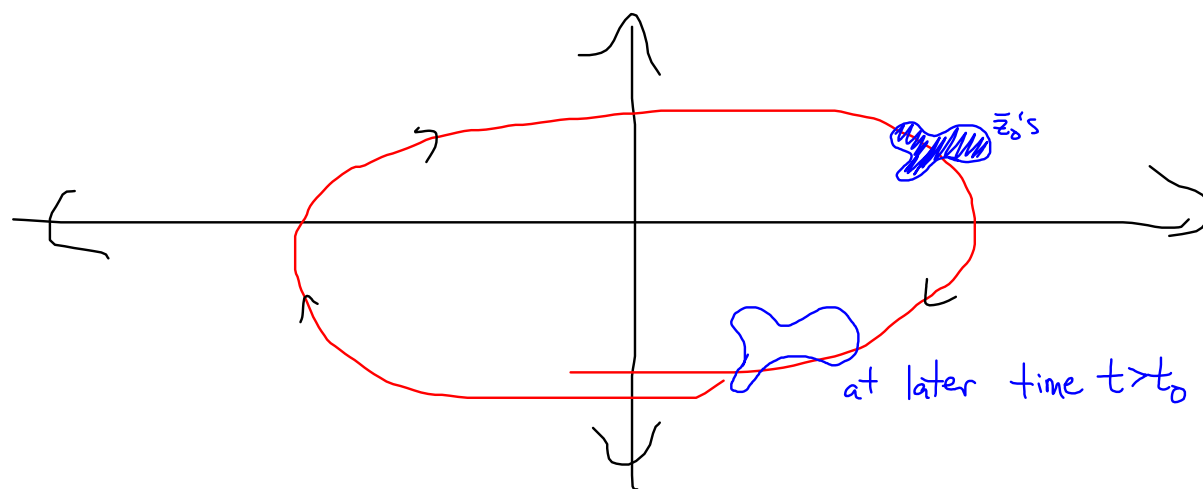
Recall we can put q 's + p 's all into one big vector $\vec{Z} = (\vec{q}, \vec{p})$

Hamilton's eq's $\rightarrow \dot{\vec{Z}} = \vec{h}(\vec{Z})$

\uparrow
 phase space velocity
 $(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i})$

Since this is a set of $2n$ first order eq's, only need one initial condition vector, \vec{Z}_0

Q. How do nearby \vec{Z}_0 's evolve as a function of time? Tuesday, we saw that they all create neighboring ellipses



Example Body in free-fall Consider dropping a ball from various heights + w/ various initial velocities

$H = T + U = \frac{p^2}{2m} - mgx$

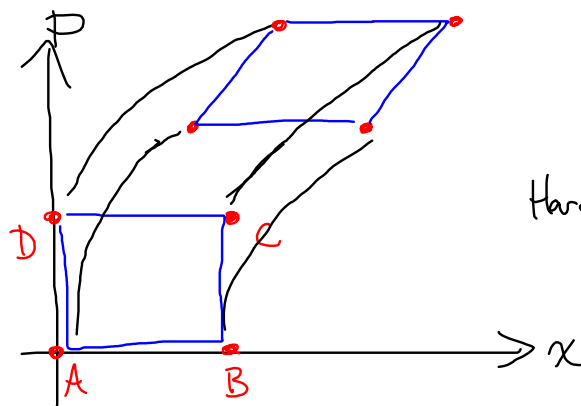
\downarrow measured downward

Initial conditions

- A. $x_0 = p_0 = 0$
- B. $x_0 = X \quad p_0 = 0$
- C. $x_0 = X \quad p_0 = P$
- D. $x_0 = 0 \quad p_0 = P$

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad p = -\frac{\partial H}{\partial x} = mg$

$x = x_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2$
 $p = p_0 + mgt$



Hard to see, but volume of \square unchanged!

Same base $A_0 B_0 = AB$
Same height

Liouville's Theorem

Conservation of phase space volume generalizes to more complicated systems

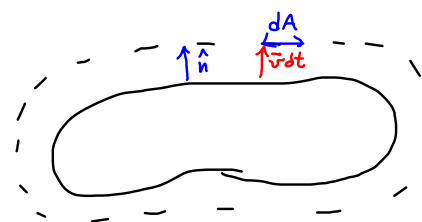
First, a couple of mathematical results

I. Change in volume over time

$$\delta V = \int_S \hat{n} \cdot \vec{v} \delta t dA$$

↑ surface integral

$$\frac{dV}{dt} = \int_S \hat{n} \cdot \vec{v} dA$$



Works for arbitrary dim.

II. Gauss' Theorem

divergence of vector field: $\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$$\int_V \vec{\nabla} \cdot \vec{v} dV = \int_S \hat{n} \cdot \vec{v} dA$$

$$\Rightarrow \frac{dV}{dt} = \int_V \vec{\nabla} \cdot \vec{v}$$

Apply this result to phase space volume

$$\vec{v} \rightarrow \frac{d}{dt} (q_1, \dots, q_n, p_1, \dots, p_n) = \dot{\vec{z}}$$

$$\vec{\nabla} \cdot \vec{v} \rightarrow \frac{\partial \dot{q}_1}{\partial q_1} + \dots + \frac{\partial \dot{q}_n}{\partial q_n} + \frac{\partial \dot{p}_1}{\partial p_1} + \dots + \frac{\partial \dot{p}_n}{\partial p_n} = \frac{\partial}{\partial q_1} \left(\frac{\partial H}{\partial p_1} \right) + \dots + \frac{\partial}{\partial p_1} \left(-\frac{\partial H}{\partial q_1} \right) + \dots$$

All cancel!
 $\vec{\nabla} \cdot \dot{\vec{z}} = 0$

$$\frac{dV_{\text{ps}}}{dt} = \int_V \vec{\nabla} \cdot \dot{\vec{z}} dV = 0$$

True generically even for non-linear systems!

Shape of volume can become very complicated (e.g. very elongated)