Lecture 13.1 - Physics 523

$\boxed{\text{Hamiltonian Mechanics}}$ In the early part of this course, we covered Lagrangian mechanics

→ Well-suited to conservative systems (esp. those with symmetries)

Advantage of Hamiltonian mechanics → easier to extend to QM, prove things about chaotic theories

Review of Lagrangian Mech. $\mathcal{L} = \mathcal{L}(\underbrace{q_1, \ldots, q_n}_{\text{generalized coordinates}}; \dot{q}_1, \ldots, \dot{q}_n; t) = T - U$ Eq. of motion → $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ → w/ initial cond., derive possible trajectories

Recall $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$ (generalized momentum) and $\frac{\partial \mathcal{L}}{\partial q_i} = F_i$ (generalized force) E-L eq.'s are just $\boxed{F_i = \dot{p}_i}$

Take total time derivative of $\mathcal{L}$: $\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\left(\sum_i p_i \dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t}$

or: $-\frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\Big[\underbrace{\sum_i p_i \dot{q}_i - \mathcal{L}}_{\mathcal{H}}\Big]$

$\mathcal{H}$ (the Hamiltonian) → We will see that we can do classical mechanics just as well with $\mathcal{H}$.

In the Hamiltonian approach, the $2n$ coordinates we use are $q_1, \ldots, q_n$ and their generalized momenta $p_1, \ldots, p_n$ "Phase Space"

(in $\mathcal{L}$ approach, we used coordinates + velocities)

Aside: the advantages of Phase space over Lagrangian "state space" lie in the geometric properties of phase space

Hamilton's Equations: Let us start with the result, which we will prove later

H's eq's are $\boxed{\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}}$ → $2n$ 1st order eq's (compare w/ n 2nd order E-L eq's)

$\boxed{\text{Example}}$ SHO from $\mathcal{L}$ + $\mathcal{H}$

$T = \frac{p^2}{2m} = \frac{1}{2}m\dot{x}^2$ $\quad U = \frac{1}{2}kx^2$

I. $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow -kx = m\ddot{x} \Rightarrow \ddot{x} = -\frac{k}{m}x$$

II. $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad \dot{p} = -kx \qquad \dot{x} = \frac{p}{m}$

note this just defines p in terms of $\dot{x}$

Easy to check that $\mathcal{H}$-formalism gives same eq.'s!

$\boxed{\text{Derivation of Hamilton's eq's (one dof} \Rightarrow \text{2D Phase Space)}}$

Starting point → conservative system w/ natural coord.

$q_i = q_i(\vec{r})$, not $q_i(\vec{r},t)$

$$\mathcal{L} = T(q,\dot{q}) - U(q) = \underbrace{\frac{1}{2}A(q)\dot{q}^2}_{\text{since } T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\overbrace{\frac{\partial \vec{r}}{\partial q}\cdot\frac{\partial \vec{r}}{\partial q}}^{A(q)}\dot{q}^2} - U(q) \qquad (\mathcal{H} = p\dot{q} - \mathcal{L})$$

now $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q}$ or $\dot{q} = \frac{p}{A(q)}$

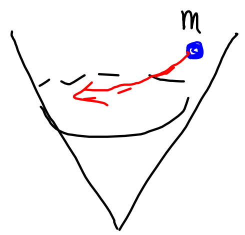
can write $\mathcal{L} = \mathcal{L}(q, \dot{q}(q,p)) \Rightarrow \mathcal{H} = p\dot{q}(q,p) - \mathcal{L}(q, \dot{q}(q,p))$

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q}(q,p) + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p} = \dot{q}$$

$$\frac{\partial \mathcal{H}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \overbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}^{p}\frac{\partial \dot{q}}{\partial q} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -\dot{p} \Rightarrow \boxed{-\frac{\partial \mathcal{H}}{\partial q} = \dot{p} \qquad \frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

See textbook for derivation for n d.o.f.

Example → mass on cone



In cylindrical coordinates, $\rho = cz$

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m(\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2) = \frac{1}{2} m((c^2+1)\dot{z}^2 + (cz\dot{\phi})^2)$$

$$U = mgz$$

} 2D system / 4D phase space

Generalized momenta: $p_z = \frac{\partial T}{\partial \dot{z}} = m(c^2+1)\dot{z}$ $\quad p_\phi = mc^2 z^2 \dot{\phi}$ (Note ϕ is ignorable $\Rightarrow$ p_ϕ conserved)

Here, system is conservative + coord's are natural $\Rightarrow$ $\mathcal{H} = T + U = \frac{p_z^2}{2m(c^2+1)} + \frac{p_\phi^2}{2m\, c^2 z^2} + mgz$

Hamilton's eq's:

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m(c^2+1)} = \dot{z} \qquad \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mc^2 z^2} = \dot{\phi}$$

$$\frac{\partial H}{\partial z} = -\frac{p_\phi^2}{mc^2 z^3} + mg = -\dot{p}_z \qquad \underbrace{\frac{\partial H}{\partial \phi}}_{\text{the cyclic coord.}} = 0 = -\dot{p}_\phi$$

the cyclic coord. $p_\phi = \text{const} = K$

Note: Here $H = H(p_z, p_\phi, z) = H(p_z, k, z) = H(p_z, z)$

→ ignorable coord. immediately reduces dimensionality!

This is an advantage of Hamiltonian formalism!

(more convoluted in $\mathcal{L}$ formalism)

Hamilton's equations w/ phase space vectors

Phase space is 2n dimensional → can use vectors in this 2n-dim space to describe motion

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = f_i(\vec{q}, \vec{p}) \Rightarrow \dot{\vec{q}} = \vec{f}(\vec{q}, \vec{p})$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = g_i(\vec{q}, \vec{p}) \Rightarrow \dot{\vec{p}} = \vec{g}(\vec{q}, \vec{p})$$

Combine into one vector eq. → $\vec{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$

$\vec{h} = (f_1(\vec{z}), \ldots, f_n(\vec{z}), g_1(\vec{z}), \ldots, g_n(\vec{z}))$

Now have $\boxed{\dot{\vec{z}} = \vec{h}(\vec{z})}$

→ In doing this, we have placed canonical coordinates + momenta on similar footing

Phase space vector $\vec{\vec{z}} = (\vec{q}, \vec{p})$ gives position in phase space $\rightarrow$ $\vec{\vec{z}}_0 = (\vec{q}_0, \vec{p}_0)$ give all conditions needed to solve 2n 1st order

$\vec{\vec{z}}(t)$ for some $\vec{\vec{z}}_0(t_0)$ is "Phase space orbit" $\rightarrow$ same info as trajectory $\vec{q}(t)$

Hard to visualize for $n \geq 2$ (4D or higher phase space)

$\boxed{\text{Example} \rightarrow \text{SHO}}$ $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \Rightarrow \dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$ $\quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x$ $\quad$ Solution is $x = A\cos(\omega t - \delta)$

$p = -m\omega A \sin(\omega t - \delta)$

$\mathcal{H}$ is conserved $\rightarrow$ $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 A^2$ or $\frac{p^2}{m^2\omega^2 A^2} + \frac{x^2}{A^2} = 1$ Ellipse!