Lecture 12.2 - Physics 523

Normal Coordinates Just as we identified 3 special axes associated w/ rigid body rotation → orthogonal set solutions to these oscillator problems define "orthogonal" solutions (normal modes)

We can define a set of coordinates in which the motion is very simply described

→ Recall our solutions for equal masses and equal k's:

$$\left.\begin{aligned}\vec{x}_1(t) &= A_1 \begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_1 t - \delta_1)\\ \vec{x}_2(t) &= A_2 \begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_2 t - \delta_2)\end{aligned}\right\}$$

define new coord's $\xi_1 = \frac{1}{2}(x_1 + x_2)$

$\xi_2 = \frac{1}{2}(x_1 - x_2)$

<u>Note</u> for $\vec{x}_1$ solution, $\xi_1 \neq 0$, $\xi_2 = 0$

$\vec{x}_1 \rightarrow \xi_1(t) = A_1 \cos(\omega_1 t - \delta_1)$ → 1st normal mode

$\vec{x}_2 \rightarrow \xi_2(t) = A_2 \cos(\omega_2 t - \delta_2)$ → 2'nd normal mode

★ This is actually a very general property of solutions to sets of linear homogenous diff. eq.'s

→ solutions form an <u>orthogonal set</u> Extremely useful → ie, fourier decomposition

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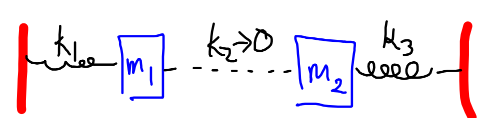
2 Weakly coupled oscillators Note that there is a limit in which the oscillators in the above system "decouple"

<u>$k_2 \rightarrow 0$</u>:

$$\hat{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \xrightarrow{k_2 \rightarrow 0} \begin{pmatrix} k_1 & 0 \\ 0 & k_3 \end{pmatrix}$$



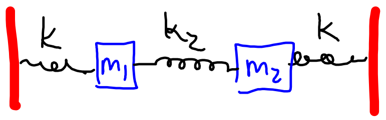
$x_1(t) = A_1 \cos\left(\sqrt{\frac{k_1}{m_1}}\, t - \delta_1\right)$

$x_2(t) = A_2 \cos\left(\sqrt{\frac{k_3}{m_2}}\, t - \delta_2\right)$

Q. In what sense can we treat k_2 as a perturbation to these simple uncoupled solutions?

Take the symmetric case $k_1 = k_3 = k$ $\hat{K} = \begin{pmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{pmatrix}$ $\hat{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$m_1 = m_2 = m$



Find freq.'s for normal modes: $\det(\hat{K} - \omega^2 \hat{M}) = \det\begin{pmatrix} k+k_2-\omega^2 m & -k_2 \\ -k_2 & k+k_2-\omega^2 m \end{pmatrix} = (k+k_2-\omega^2 m)^2 - k_2^2$

$$= m^2\left[\omega^4 - 2\omega^2\left(\frac{k+k_2}{m}\right) + \frac{k^2+2kk_2}{m^2}\right]$$

$$= m^2\left(\omega^2 - k/m\right)\left(\omega^2 - \frac{k+2k_2}{m}\right)$$

Frequencies $\omega_1 = \sqrt{\frac{k}{m}}$ $\rightarrow$ the one that doesn't even activate k_2 spring (oscillations in phase)

$\omega_2 = \sqrt{\frac{k+2k_2}{m}}$ $\rightarrow$ out of phase oscillations $\omega_2 \approx \omega_1$ for small k_2

Define $\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)$ $\rightarrow$ $\omega_1 = \omega_0 - \varepsilon$, $\omega_2 = \omega_0 + \varepsilon$

Solution $\rightarrow$ $\vec{x}_1(t) = x_1^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(\omega_0 - \varepsilon)t}$ $\vec{x}_2(t) = x_2^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i(\omega_0+\varepsilon)t}$

$$\vec{X}(t) = \left[x_1^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\varepsilon t} + x_2^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{+i\varepsilon t}\right] e^{i\omega_0 t}$$

phase shift between solutions slowly varies with time

What does this actually look like? Take $x_1^0 = x_2^0 = \text{real} = \frac{A}{2}$ $\vec{X}(t) = Ae^{i\omega_0 t}\begin{pmatrix} \cos \varepsilon t \\ -i \sin \varepsilon t \end{pmatrix} \xrightarrow{\mathbb{R}\text{ part}} \begin{array}{l} x_1 = A \cos \omega_0 t \cos \varepsilon t \\ x_2 = A \sin \omega_0 t \sin \varepsilon t \end{array}$

x_2 starts out zero, and starts growing in amplitude over time, w/ x_1 shrinking in amplitude

SEE Fig 11.8

Recall our earlier statements → any 1D system sufficiently near min of potential energy behaves like a mass on a spring

For systems w/ N generalized coordinates → any system w/ N dof near a minimum of the potential energy behaves like N coupled oscillators

N linear diff. eq.'s $\hat{M}\cdot\ddot{\vec{q}} = -\hat{K}\cdot\vec{q}$ ($\hat{M}$ & $\hat{K}$ not necessarily straightford masses / spring constants)

Example: The double pendulum. Let's start at the level of the equations of motion (your HW problem from awhile ago)

For small oscillations (near stationary point)

$$(m_1+m_2)L_1^2\ddot{\phi}_1 + m_2L_1L_2\ddot{\phi}_2 = -(m_1+m_2)gL_1\phi_1$$
$$m_2L_1L_2\ddot{\phi}_1 + m_2L_2^2\ddot{\phi}_2 = -m_2gL_2\phi_2$$

$$\hat{M} = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix} \qquad \hat{K} = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}$$

$$\det(\hat{K}-\omega^2\hat{M}) = \det\begin{pmatrix} (m_1+m_2)(gL_1-\omega^2L_1^2) & -\omega^2m_2L_1L_2 \\ -\omega^2m_2L_1L_2 & m_2(gL_2-\omega^2L_2^2) \end{pmatrix}$$

Equal masses & L's

$$\det\begin{pmatrix} 2mL^2(\frac{g}{L}-\omega^2) & -\omega^2mL^2 \\ -\omega^2mL^2 & mL^2(\frac{g}{L}-\omega^2) \end{pmatrix} = 2m^2L^4(g/L-\omega^2)^2 - \omega^4m^2L^4 = 0$$

$$\omega^4 - 4\frac{g}{L}\omega^2 + 2\left(\frac{g}{L}\right)^2 = 0 \Rightarrow \omega^2 = \frac{4g/L \pm (16(g/L)^2 - 8(g/L)^2)^{1/2}}{2} = 2\frac{g}{L} \pm \sqrt{2}\frac{g}{L} = \frac{g}{L}(2\pm\sqrt{2})$$

Now to find the normal modes

$$\begin{pmatrix} 2mL^2(\frac{g}{L}-\frac{g}{L}(2\pm\sqrt{2})) & -mL^2\frac{g}{L}(2\pm\sqrt{2}) \\ -mL^2\frac{g}{L}(2\pm\sqrt{2}) & mL^2(\frac{g}{L}-\frac{g}{L}(2\pm\sqrt{2})) \end{pmatrix} = -mL^2\frac{g}{L}\begin{pmatrix} 2(1\pm\sqrt{2}) & (2\pm\sqrt{2}) \\ (2\pm\sqrt{2}) & (1\pm\sqrt{2}) \end{pmatrix}$$

$$= -mL^2\frac{g}{L}(1\pm\sqrt{2})\begin{pmatrix} 2 & \pm\sqrt{2} \\ \pm\sqrt{2} & 1 \end{pmatrix} \rightarrow \text{eigenvectors} \quad \begin{pmatrix} 2 & \pm\sqrt{2} \\ \pm\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} 2\phi_1^0 \mp\sqrt{2}\phi_2^0 \\ \mp\sqrt{2}\phi_1^0 + \phi_2^0 \end{pmatrix} = 0$$

$$2\phi_1^0 = \mp\sqrt{2}\phi_2^0 \qquad \vec{\phi}^0 = \begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}$$

In Summary our canonical coordinates are ϕ_1 + ϕ_2, we go to small angle approx. (coincides w/ global min of $U(\phi_1,\phi_2)$)

→ eom are then linear, can fit into $\hat{M}\cdot\ddot{\vec{q}} = -\hat{K}\cdot\vec{q}$ ⇒ same kind of eigenvalue problem!

Normal Modes I. $\omega_1 = \frac{g}{L}(2+\sqrt{2})$ $\vec{\phi}_0^{(1)} = A_1\begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}$ II. $\omega_2 = \frac{g}{L}(2-\sqrt{2})$ $\vec{\phi}_0^{(2)} = \begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$

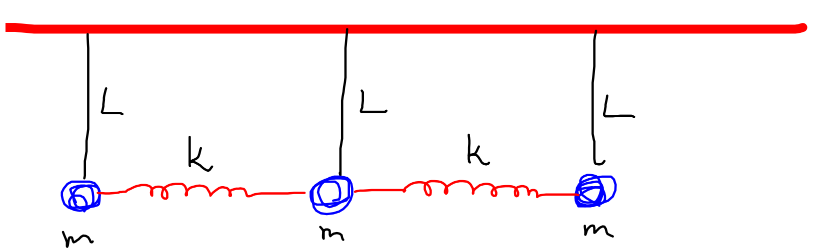
$\vec{\phi}^{(1)}(t) = A_1\begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\cos(\omega_1 t - \delta_1)$ $\vec{\phi}^{(2)}(t) = A_2\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}\cos(\omega_2 t - \delta_2)$

"out of phase" "in phase"

$$\boxed{\vec{\phi}(t) = A_1\begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\cos(\omega_1 t - \delta_1) + A_2\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}\cos(\omega_2 t - \delta_2)}$$

Normal coordinates $\xi_\pm = \frac{1}{\sqrt{3}}(\phi_1 \pm \sqrt{2}\,\phi_2)$ → directions of extrema of ascent/descent of $U(q_1,q_2)$

A 3D system 3 coupled pendula



$T = \frac{1}{2}mL^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2)$

$U_g = \frac{1}{2}mgL(\phi_1^2 + \phi_2^2 + \phi_3^2)$

$U_{spring} = \underbrace{\frac{1}{2}kL^2\left[(\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2\right]}_{\text{coupling terms!}} = \frac{1}{2}kL^2\left[\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3\right]$

$\hat{M} = mL^2\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}$ $\hat{K} = mgL\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix} + kL^2\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}$

At this point, it's time to work out some yucky algebra

3 ω's + 3 $\vec{\Phi}_0$'s for 3 normal modes