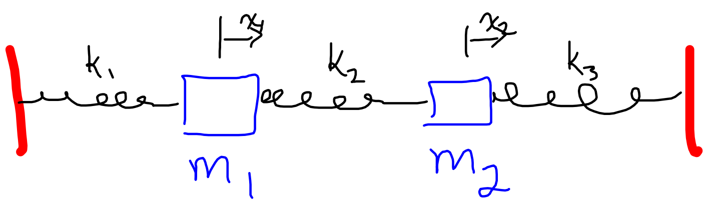
Lecture 11.2 - Physics 523

Quick review of coupled oscillators

An overture for coupled oscillators

Many many systems can be modeled as masses connected by springs

How do we approach the study of such systems?



Equations of motion?

$$F_1 = -k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1 \qquad m_1 \ddot{x}_1 = -(k_1+k_2)x_1 - (-k_2)x_2$$

$$F_2 = -k_3 x_2 - k_2(x_2 - x_1) = m_2 \ddot{x}_2 \qquad m_2 \ddot{x}_2 = -(-k_2)x_1 - (k_2+k_3)x_2$$

Can rewrite this in matrix form $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\boxed{\hat{M}\cdot\ddot{\vec{x}} = -\hat{K}\cdot\vec{x}}$

Our intuition & experience tells us there should be solutions where the system oscillates as a whole, w/ a single frequency

Search for these! $\vec{x} = \vec{x}_0 e^{i\omega t} \rightarrow -\omega^2 \hat{M}\cdot\vec{x}_0 = -\hat{K}\cdot\vec{x}_0$ or $(\hat{K} - \omega^2\hat{M})\cdot\vec{x}_0 = 0$

this should look somewhat familiar! For non-trivial $\vec{x}_0$, require $\det(\hat{K} - \omega^2\hat{M}) = 0$ Solve for ω!

Reduces to an eigenvalue problem!! $\rightarrow$ 2 solutions, for this system in general

Let us analyze some special cases equal masses + equal k's $\hat{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ $\hat{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$

$$\det(\hat{K} - \omega^2 \hat{M}) = (2k - \omega^2 m)^2 - k^2 = 3k^2 - 4\omega^2 mk + \omega^4 m^2 = 0 \qquad \left(\omega^2 - 3\tfrac{k}{m}\right)\left(\omega^2 - k/m\right) = 0$$

2 frequencies $\boxed{\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}}$

So what is the motion associated with these 2 frequencies? Need to solve the linear equations $(K - \omega^2 M)\vec{x}_0 = 0$ for ω_1, ω_2

$\omega_1 = \sqrt{k/m}$ $\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \frac{k}{m}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$ Need $\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} = 0$, $x_0^1 = -x_0^2$ (again, only 1 eq. since det $= 0 \Rightarrow$ linearly dep eqs)

eigenvectors

solution $\vec{x}(t) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t - \delta)$

→ carts move back + forth in synchrony

$\omega_2 = \sqrt{\frac{3k}{m}}$ $\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \frac{3k}{m}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}$ Need $\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = 0$, or $x_1^0 + x_2^0 = 0$

Solution $\vec{x}(t) = A\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t - \delta)$

→ carts' motion is mirror image of each other

Full Solution $\boxed{\vec{x}(t) = A_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t - \delta_1) + A_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t - \delta_2)}$

→ while motion of each frequency on its own is simple, motion of superposition can be very complicated

Example: take $k_1 = k_3 = 2k$, $k_2 = k \Rightarrow \hat{K} = \begin{pmatrix} 3k & -k \\ -k & 3k \end{pmatrix}$ $(3k-\omega^2 m)^2 - k^2 = 0 \Rightarrow m^2\left(\omega^4 - 6\frac{k}{m}\omega^2 + 8\frac{k^2}{m^2}\right) \Rightarrow \left(\omega^2 - 4\frac{k}{m}\right)\left(\omega^2 - 2\frac{k}{m}\right) = 0$

$$\boxed{\omega_1 = 2\sqrt{\frac{k}{m}} \quad \omega_2 = \sqrt{\frac{2k}{m}}}$$

$(\hat{K} - \omega_{1,2}\hat{M})\vec{x}^{\,0}_{1,2} = 0$

$$\begin{pmatrix} 3k & -k \\ -k & 3k \end{pmatrix} - 4\frac{k}{m}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}$$

$$\begin{pmatrix} 3k & -k \\ -k & 3k \end{pmatrix} - 2\frac{k}{m}\begin{pmatrix} m & \\ & m \end{pmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

$\omega_1 \to \vec{x}_1(t) = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_1 t - \delta_1)$

$\omega_2 \to \vec{x}_2(t) = A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$

Note that qualitative motion is the same! (reflection symmetry)

Normal Coordinates Just as we identified 3 special axes associated w/ rigid body rotation → orthogonal set solutions to these oscillator problems define "orthogonal" solutions (normal modes)

We can define a set of coordinates in which the motion is very simply described

→ Recall our solutions for equal masses and equal k's: $\vec{x}_1(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1)$, $\vec{x}_2(t) = A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$ } define new coord's $\xi_1 = \frac{1}{2}(x_1 + x_2)$, $\xi_2 = \frac{1}{2}(x_1 - x_2)$

Note for $\vec{x}_1$ solution, $\xi_1 \neq 0$, $\xi_2 = 0$ $\quad \vec{x}_1 \to \xi_1(t) = A_1 \cos(\omega_1 t - \delta_1) \to$ 1st normal mode

$\vec{x}_2 \to \xi_2(t) = A_2 \cos(\omega_2 t - \delta_2) \to$ 2'nd normal mode

★ This is actually a very general property of solutions to sets of linear homogenous diff. eq.'s

→ solutions form an orthogonal set Extremely useful → ie, fourier decomposition

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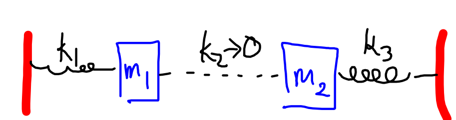
2 Weakly coupled oscillators Note that there is a limit in which the oscillators in the above system "decouple"

$k_2 \to 0$: $\hat{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \xrightarrow{k_2 \to 0} \begin{pmatrix} k_1 & 0 \\ 0 & k_3 \end{pmatrix}$



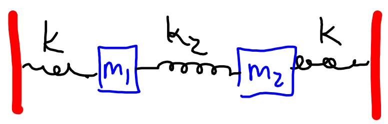
$x_1(t) = A_1 \cos\left(\sqrt{\frac{k_1}{m_1}}\, t - \delta_1\right)$

$x_2(t) = A_2 \cos\left(\sqrt{\frac{k_3}{m_2}}\, t - \delta_2\right)$

Q. In what sense can we treat k_2 as a perturbation to these simple uncoupled solutions?

Take the symmetric case $k_1 = k_3 = k$ $\quad$ $\hat{K} = \begin{pmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{pmatrix}$ $\quad$ $\hat{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$m_1 = m_2 = m$



Find freq.'s for normal modes: $\det(\hat{K} - \omega^2 \hat{M}) = \det\begin{pmatrix} k+k_2-\omega^2 m & -k_2 \\ -k_2 & k+k_2-\omega^2 m \end{pmatrix} = (k+k_2-\omega^2 m)^2 - k_2^2$

$$= m^2\left[\omega^4 - 2\omega^2\left(\frac{k+k_2}{m}\right) + \frac{k^2+2kk_2}{m^2}\right]$$

$$= m^2\left(\omega^2 - k/m\right)\left(\omega^2 - \frac{k+2k_2}{m}\right)$$

Frequencies $\quad \omega_1 = \sqrt{\frac{k}{m}} \rightarrow$ the one that doesn't even activate k_2 spring (oscillations in phase)

$\omega_2 = \sqrt{\frac{k+2k_2}{m}} \rightarrow$ out of phase oscillations $\quad \omega_2 \approx \omega_1$ for small k_2

Define $\quad \omega_0 = \frac{1}{2}(\omega_1 + \omega_2) \rightarrow \begin{array}{l} \omega_1 = \omega_0 - \varepsilon \\ \omega_2 = \omega_0 + \varepsilon \end{array}$ $\quad$ Solution $\rightarrow$ $\vec{x}_1(t) = x_1^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(\omega_0 - \varepsilon)t}$ $\quad$ $\vec{x}_2(t) = x_2^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i(\omega_0+\varepsilon)t}$

$$\vec{X}(t) = \left[x_1^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\varepsilon t} + x_2^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{+i\varepsilon t}\right] e^{i\omega_0 t}$$

phase shift between solutions
slowly varies with time

What does this actually look like? Take $x_1^0 = x_2^0 = \text{real} = \frac{A}{2}$ $\quad$ $\vec{X}(t) = A e^{i\omega_0 t} \begin{pmatrix} \cos \varepsilon t \\ -i \sin \varepsilon t \end{pmatrix} \xrightarrow{\mathbb{R}\text{ part}} \begin{array}{l} x_1 = A \cos \omega_0 t \cos \varepsilon t \\ x_2 = A \sin \omega_0 t \sin \varepsilon t \end{array}$

x_2 starts out zero, and starts growing in amplitude over time, w/ x_1 shrinking in amplitude

SEE Fig 11.8