Lecture 11.1 - Physics 523

Euler's Equations

Euler's Eq's → Equations specifying evolution in rotating principal axis frame

Body frame vs space frame: Body frame → axes coincide w/ principal axes of rigid body (rotating/non-inertial in general)

In the body frame coordinates, we have $\vec{L} = (\underbrace{\lambda_1, \lambda_2, \lambda_3}_{\text{principal moments}}\text{: } \lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$ basis is $\hat{e}_1, \hat{e}_2, \hat{e}_3$ (principal axes)

We want to describe time evolution of $\vec{L}$ in an inertial "space frame" $\hat{x}, \hat{y}, \hat{z}$ $\quad \left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\tau}$ (usual relation)

But we want this in the non-inertial frame, $\left(\frac{d\vec{Q}}{dt}\right)_{space} = \left(\frac{d\vec{Q}}{dt}\right)_{body} + \vec{\omega}\times\vec{Q} \Rightarrow \left(\frac{d\vec{L}}{dt}\right)_{space} = \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\tau}$

component by component $\quad \dot{\vec{L}} \to \begin{pmatrix}\lambda_1\dot{\omega}_1 \\ \lambda_2\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3\end{pmatrix} \quad \vec{\omega}\times\vec{L} = \det\begin{pmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3\end{pmatrix} = \hat{e}_1\omega_2\omega_3(\lambda_3-\lambda_2) + \hat{e}_2\omega_1\omega_3(\lambda_1-\lambda_3) + \hat{e}_3\omega_1\omega_2(\lambda_2-\lambda_1)$

3 Equations

$$\boxed{\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3 &= \tau_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_1\omega_3 &= \tau_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2 &= \tau_3 \end{aligned}}$$

"Euler's Equations"

Mostly useful when $\vec{\tau} = 0$, since $\vec{\tau}$ often complicated in the non-inertial frame

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Solutions to Euler's Equations w/ $\vec{\tau} = 0$

$$\left.\begin{aligned} \lambda_1\dot{\omega}_1 &= (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 &= (\lambda_3-\lambda_1)\omega_1\omega_3 \\ \lambda_3\dot{\omega}_3 &= (\lambda_1-\lambda_2)\omega_1\omega_2 \end{aligned}\right\}$$

just components of $\dot{\vec{L}} = \vec{L}\times\vec{\omega}$ in "body frame"

Case I Rotation about a principal axis $\rightarrow$ If rotation is about some given principal axis (1, 2, or 3) then other ω's are $\varnothing$

Euler's equations: $\lambda_1\dot{\omega}_1 = \lambda_2\dot{\omega}_2 = \lambda_3\dot{\omega}_3 = 0$ Constant rotation $\dot{\vec{\omega}} = 0$

Case II Rotation about some non-principal axis $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ 3 non-trivial coupled equations! $\dot{\vec{\omega}} \neq 0$

Evolution of $\vec{\omega}$ Recall what a thrown football looks like, usually there is a "wobble" of the main axis of the football unless it is a very clean throw. How do we describe this motion?

Stability of $\vec{\omega}$ Say we initially have large ω_3, but small ω_1, ω_2

ω_3 eq.: $\lambda_3\dot{\omega}_3 = (\lambda_1-\lambda_2)\,\omega_1\omega_2 \simeq 0$ (2 small #s multiplied)

$\Rightarrow \dot{\omega}_3 \simeq 0 \Rightarrow \omega_3 = \text{const.}$

$\omega_1 + \omega_2$ eq's

$$\dot{\omega}_1 = \left[\frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\right]\omega_2$$

$$\dot{\omega}_2 = \left[\frac{\lambda_3-\lambda_1}{\lambda_2}\omega_3\right]\omega_1$$

Trick You can often combine lower order diff. eq.'s for higher order ones

$$\dot{\omega}_1 = \left[\frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\right]\omega_2 \Rightarrow \ddot{\omega}_1 = \left[\frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\right]\dot{\omega}_2 = \left[\frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\right]\left[\frac{\lambda_3-\lambda_1}{\lambda_2}\omega_3\right]\omega_1 = -\underbrace{\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]}_{\text{Dynamics depends strongly on the sign of this piece}}\omega_1$$

Now 2nd order uncoupled!

I. If λ_3 is either the smallest or largest principal moment, solutions are oscillatory (sin + cos)

II If λ_3 is the intermediate eigenvalue of $\hat{I}$ (principal moment) then solutions are exponential damping + growth

exponential growth $\Rightarrow$ small $\omega_1 + \omega_2$ quickly grow to large deviations

Stability for rotation about extremal axes.

Example 1) Show that Euler's Eq. show that for $\vec{\tau}=0$, $\frac{d}{dt}|\vec{L}|=0$

$\dot{\vec{L}} = \vec{L}\times\vec{\omega}$ $\quad |\vec{L}| = [\vec{L}\cdot\vec{L}]^{1/2} \Rightarrow \frac{d}{dt}|\vec{L}| = \frac{\dot{\vec{L}}\cdot\vec{L}}{|\vec{L}|} = \frac{\vec{L}\cdot(\vec{L}\times\vec{\omega})}{\vec{L}} = 0 \Rightarrow |\vec{L}|$ is const.

ϕ since $\vec{L} \perp$ to $(\vec{\omega}\times\vec{L})$

2) Show that $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\hat{I}\cdot\vec{\omega}$ is constant

$\dot{T} = \frac{1}{2}\dot{\vec{\omega}}\cdot\hat{I}\cdot\vec{\omega} + \frac{1}{2}\vec{\omega}\cdot\hat{I}\cdot\dot{\vec{\omega}} = \frac{1}{2}\dot{\vec{L}}\cdot\vec{\omega} + \frac{1}{2}\vec{\omega}\cdot\dot{\vec{L}} = \frac{1}{2}(\vec{\omega}\times\vec{L})\cdot\vec{\omega} + \frac{1}{2}\vec{\omega}\cdot(\vec{\omega}\times\vec{L})$

both zero for same reason

So if λ_3 is an extreme P.Moment, have stable rotation → what if 2 P.M.'s are equal? E.g. axis of symmetry → football / top

$\boxed{\text{2 equal moments → free precession}}$

Euler's eq's $w/\vec{\tau}=0$ simplify

$$\left.\begin{aligned} \lambda_1\dot{\omega}_1 &= (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 &= (\lambda_3-\lambda_1)\omega_1\omega_3 \\ \lambda_3\dot{\omega}_3 &= (\lambda_1-\lambda_2)\omega_1\omega_2 \end{aligned}\right\} \ w/\ \lambda_1=\lambda_2 \quad \begin{aligned} \lambda\dot{\omega}_1 &= (\lambda-\lambda_3)\omega_2\omega_3 \\ \lambda\dot{\omega}_2 &= (\lambda_3-\lambda)\omega_1\omega_3 \\ \boxed{\lambda_3\dot{\omega}_3} &\boxed{= \phi} \end{aligned}$$

true for all rotations

Re-write as $\quad \dot{\omega}_1 = \Omega_b\omega_2 \quad \dot{\omega}_2 = -\Omega_b\omega_1 \quad \Omega_b = \left(\frac{\lambda-\lambda_3}{\lambda}\right)\omega_3$

to solve this, we use that tried + true complexification trick

$\eta \equiv \omega_1 + i\omega_2 \qquad \underbrace{\dot{\eta} = -i\Omega_b\eta}_{\text{sol'n } \eta = \eta_0 e^{-i\Omega_b t}} \rightarrow \dot{\omega}_1 + i\dot{\omega}_2 = -i\Omega_b(\omega_1+i\omega_2) = \Omega_b\omega_2 - i\Omega_b\omega_1$ ✓ works

$\vec{\omega} = (\omega_0\cos(\Omega_b t-\delta), -\omega_0\sin(\Omega_b t-\delta), \omega_3)$

$\vec{L} = (\lambda\,\omega_0\cos(\Omega_b t-\delta), -\lambda\omega_0\sin(\Omega_b t-\delta), \lambda_3\omega_3)$

so $|\vec{\omega}|$ = const $|\vec{L}|$ = const, + angles also constant. (between $\hat{e}_3$ as well)

This has all been done in the body frame (non-inertial) In the "space" (inertial) frame, $\vec{L}$ = const. and $\vec{\omega}\,/\,\hat{e}_3$ precess about fixed $\vec{L}$ axis about fixed $\vec{L}$ axis

Can be shown: $\Omega_3 = \frac{|\vec{L}|}{\lambda}$

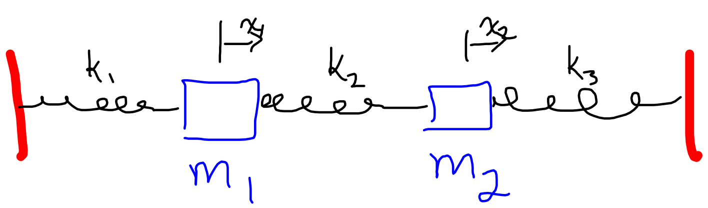
Moral whenever an axially symmetric body rotates w/ $\vec{\omega}$, $\vec{\omega}$ precesses about fixed $\vec{L}$ axis

$\rightarrow$ In fact, the earth not only precesses due to $\vec{\tau}$ from sun/moon, but also because $\vec{\omega}$ is not along principle moment!

An overture for coupled oscillators

Many many systems can be modeled as masses connected by springs

How do we approach the study of such systems?



Equations of motion?

$F_1 = -k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1$ $\qquad m_1 \ddot{x}_1 = -(k_1+k_2)x_1 - (-k_2)x_2$

$F_2 = -k_3 x_2 - k_2(x_2 - x_1) = m_2 \ddot{x}_2$ $\qquad m_2 \ddot{x}_2 = -(-k_2)x_1 - (k_2+k_3)x_2$

Can rewrite this in matrix form $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\qquad \boxed{\hat{M}\cdot\ddot{\vec{x}} = -\hat{K}\cdot\vec{x}}$

Our intuition & experience tells us there should be solutions where the system oscillates as a whole, w/ a single frequency

Search for these! $\vec{x} = \vec{x}_0 e^{i\omega t} \rightarrow -\omega^2 \hat{M}\cdot\vec{x}_0 = -\hat{K}\cdot\vec{x}_0$ or $(\hat{K} - \omega^2\hat{M})\cdot\vec{x}_0 = 0$

this should look somewhat familiar! For non-trivial $\vec{x}_0$, require $\det(\hat{K} - \omega^2\hat{M}) = 0$ Solve for ω!

Reduces to an eigenvalue problem!! $\rightarrow$ 2 solutions, for this system in general