Lecture 10.2 - Physics 523

Rotational Motion - Review

Recall: Rotational motion of rigid bodies: $\vec{L} = \hat{I} \cdot \vec{\omega}$

$\uparrow$ inertia tensor

Example: we saw that for a cube, rotating about any axis through one of its corners,
w/ sides a, mass M

$$\vec{L}_{cube} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

remember we also noticed that for $\underbrace{\vec{\omega} = \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{corner to corner axis of rotation}}$, that $\vec{L}$ was $\parallel$ to $\vec{\omega}$

All rigid bodies have 3 $\perp$ axes about which rotation produces $\vec{L} = I^i \vec{\omega}^i$

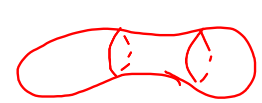
$\uparrow$ principal moments

Example $I^{c2c} \rightarrow \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{Ma^2}{6} \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow I^{c2c} = \frac{Ma^2}{6}$

If you choose the right coordinate system, $\hat{I}$ is diagonal $\hat{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

Q. How do we find this special coordinate system, and these principal moments?

$A_1 \rightarrow$ if a body is rotationally symmetric about some axis, then that axis is a principal axis

But what if we have no such guide?  (no axis of symmetry)

For any rigid body, and any fixed point O about which the body rotates, there are 3 $\perp$ axes such that $\hat{I}$ in the coordinate system spanned by those axes is diagonal.

To find the axes + principal moments, we must solve an eigenvalue / eigen vector problem. $\vec{L} = \lambda_{1,2,3}\vec{\omega}_{1,2,3}$

First, find principal moments: found via solutions to $\det[\hat{I} - \mathbb{1}\lambda] = 0$ $\rightarrow$ cubic eq., yield's at most 3 λ's as solutions

Then, find principal axes: found via solutions to sets of linear eq.'s $(\hat{I} - \lambda_{1,2,3}\mathbb{1})\vec{\omega} = 0$ each 3 eqs for 3 unknowns

Example the same cube $\hat{I} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$ $\det(\hat{I} - \lambda\mathbb{1}) = \det\left[\frac{Ma^2}{12}\begin{pmatrix} 8-\tilde{\lambda} & -3 & -3 \\ -3 & 8-\tilde{\lambda} & -3 \\ -3 & -3 & 8-\tilde{\lambda} \end{pmatrix}\right]$ $\lambda \equiv \tilde{\lambda}\,\frac{Ma^2}{12}$

$$= \left(\frac{Ma^2}{12}\right)^3\left[(8-\lambda)^3 - 54 + 27(8-\tilde{\lambda})\right] = -\left(\frac{Ma^2}{12}\right)^3(\tilde{\lambda}-11)^2(\tilde{\lambda}-2)$$

So the principal moments are $\tilde{\lambda} = 11$, $\tilde{\lambda} = 2$ or $\boxed{\lambda = \frac{Ma^2}{12}\cdot 11 \text{ or } \lambda = \frac{Ma^2}{12}\cdot 2}$ Note there are only 2 distinct eigenvalues (principal moments)

Now to find the principal axes: Need to solve 2 sets of 3 eq's

I. $(\hat{I} - 2\frac{Ma^2}{12}\mathbb{1})\vec{\omega} = 0$

$$\frac{Ma^2}{12}\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$2\omega_x - \omega_y - \omega_z = 0$

$2\omega_y - \omega_x - \omega_z = 0$

$2\omega_z - \omega_x - \omega_y = 0$

Sol. $\omega_x = \omega_y = \omega_z$ $\vec{\omega} = \frac{1}{\sqrt{3}}(1,1,1)$

II. $(\hat{I} - 11\frac{Ma^2}{12}\mathbb{1})\vec{\omega}$

$$\frac{Ma^2}{12}\begin{pmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

all 3 same eq.! $\omega_x + \omega_y + \omega_z = 0$

(any 2 vectors $\perp$ to one for $\tilde{\lambda} = 2$)

But this was a special case w/ 2 degenerate eigenvalues → When all 3 are distinct, then all 3 eigenvectors / principal axes are completely determined by this procedure

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Examples $\hat{I} = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$

$\det(\hat{I} - \lambda \mathbb{1}) = \det \begin{pmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & 1 \\ 0 & 1 & 6-\lambda \end{pmatrix} = (10-\lambda)(6-\lambda)^2 - (10-\lambda) = (10-\lambda)[35 - 12\lambda - \lambda^2]$

$= -(10-\lambda)(\lambda-7)(\lambda-5)$

eigenvalues are 10, 7, 5, P.M.'s are $\boxed{10ma^2,\ 7ma^2,\ 5ma^2}$

Principal axes

$\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} - \lambda \mathbb{1}\right]\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \quad \left.\begin{array}{l} -4\omega_y + \omega_z = 0 \\ \omega_y - 4\omega_z = 0 \end{array}\right\}$ both $\omega_y, \omega_z = 0$ $\quad \vec{\omega}_{(10)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

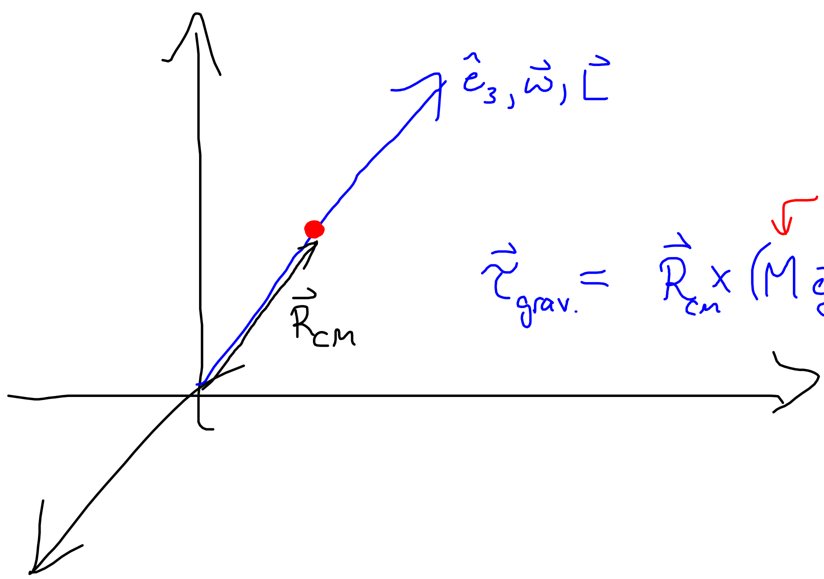
$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \quad \left.\begin{array}{l} 3\omega_x = 0 \\ \omega_y - \omega_z = 0 \end{array}\right\} \quad \vec{\omega}_{(7)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \quad \left.\begin{array}{l} 5\omega_x = 0 \\ \omega_y + \omega_z = 0 \end{array}\right\} \quad \vec{\omega}_{(5)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Precession of a top due to torque

Top is axially symmetric, so $\hat{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ where $\hat{e}_3$-axis is axis of symmetry

Without gravity, $\vec{L} = \lambda_3 \vec{\omega}$, w/ $\vec{\omega}$ along axis of top $(\hat{e}_3)$



$\vec{\tau}_{grav.} = \vec{R}_{cm} \times (M\vec{g})$ ← mass of top

Now eom states $\dot{\vec{L}} = \vec{\tau}$ → $\perp$ to $\vec{L}$!

Changes in direction, not magnitude

$\vec{L} = \lambda_3 \omega \hat{e}_3$, $\dot{\vec{L}} = \lambda_3 \omega \dot{\hat{e}}_3 = -R_{cm} M g\, \hat{e}_3 \times \hat{z}$

$\dot{\vec{e}}_3 = \left(\frac{R_{CM} M g}{\lambda_3 \omega}\right) \hat{z} \times \hat{e}_3 \equiv \vec{\Omega} \times \hat{e}_3$

$\dot{\hat{e}}_3 = \vec{\Omega} \times \hat{e}_3$ → $\hat{e}_3$ axis rotates about z-axis w/ angular velocity $\boxed{\vec{\Omega} = \frac{M g R_{cm}}{\lambda_3 \omega} \hat{z}}$