HW#7 Solutions

1) In CM coordinates $\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \sum_\alpha m_\alpha [\vec{\omega}\, (\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha (\vec{\omega} \cdot \vec{r}_\alpha)]$

using a standard double x-product identity

In component form: $L_i = \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{k\alpha}^2 - x_{i\alpha} \sum_k \omega_k x_{k\alpha} \right) = \sum_{\alpha,k} \left(m_\alpha x_{k\alpha}^2 \omega_i - x_{i\alpha} x_{k\alpha} \omega_k \right)$

E.g. $L_1 = \omega_1 \left(\sum_\alpha m_\alpha (x_{1\alpha}^2 + x_{2\alpha}^2 + x_{3\alpha}^2) - m_\alpha x_{1\alpha}^2 \right) - \sum_\alpha m_\alpha x_{\alpha 1} (x_{\alpha 2} \omega_2 + x_{\alpha 3} \omega_3) = \omega_1 \sum_\alpha m_\alpha (x_{2\alpha}^2 + x_{3\alpha}^2) - \omega_2 \sum_\alpha m_\alpha x_{1\alpha} x_{2\alpha} - \omega_3 \sum_\alpha m_\alpha x_{1\alpha} x_{3\alpha}$

$= \sum_k I_{1k} \omega_k$

Generally, $\boxed{L_i = \sum_k I_{ik} \omega_k}$

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2) We have $m \left(\frac{d^2 \vec{r}}{dt^2} \right)_{S_0} = \vec{F}$ in the inertial frame and $\left(\frac{d\vec{r}}{dt} \right)_{S_0} = \left(\frac{d\vec{r}}{dt} \right)_S + \vec{\Omega} \times \vec{r}$ in non-inertial

$$\left(\frac{d^2 \vec{r}}{dt^2} \right)_{S_0} = \left(\frac{d}{dt} \right)_S \left[\left(\frac{d\vec{r}}{dt} \right)_S + \vec{\Omega} \times \vec{r} \right] + \vec{\Omega} \times \left[\left(\frac{d\vec{r}}{dt} \right)_S + \vec{\Omega} \times \vec{r} \right]$$

$$= \left(\frac{d^2 \vec{r}}{dt^2} \right)_S + \left(\frac{d\vec{\Omega}}{dt} \right)_S \times \vec{r} + 2\vec{\Omega} \times \left(\frac{d\vec{r}}{dt} \right)_S + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

In N.I. frame: $m\ddot{\vec{r}} = \vec{F} + \underbrace{m \vec{r} \times \dot{\vec{\Omega}}}_{\text{azimuthal force term!}} + 2m\vec{v} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$

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3) If coriolis + azimuthal forces are vanishing, then $m\ddot{\vec{r}} = \vec{F} + m(\vec{\Omega} \times (\vec{r} \times \vec{\Omega})) = -mg_0 \hat{r} + m\Omega^2 r \sin\theta\, \hat{\rho}$

free fall at N.P.

unit vector away from axis (as in cylindrical coord's)

$\ddot{\vec{r}} = -g_0 \left[\hat{r} - \frac{m\Omega^2 r}{g_0} \sin\theta\, \hat{\rho} \right]$ At equator, $\hat{r} = \hat{\rho}$, and $\ddot{\vec{r}} = -g_0 \hat{r} \underbrace{\left[1 - \frac{m\Omega^2 r}{g_0} \right]}_{\lambda}$

generally $\ddot{\vec{r}} = -g_0 [\hat{r} + (\lambda - 1) \sin\theta\, \hat{\rho}] = -g_0 [\hat{r} + (\lambda - 1) \sin\theta\, (\sin\theta\, \hat{r} + \cos\theta\, \hat{\theta})]$

$$= -g_0 [(1 + (\lambda - 1)\sin^2\theta)\, \hat{r} + (\lambda - 1) \sin\theta \cos\theta\, \hat{r}]$$

Taking the magnitude: $g(\theta) = |\ddot{\vec{r}}| = g_0 \left[(1 + (\lambda - 1)\sin^2\theta)^2 + (\lambda - 1)^2 \sin^2\theta \cos^2\theta \right]^{1/2}$

Simplifies to: $\boxed{g(\theta) = \frac{g_0}{\sqrt{2}} \left[1 + \lambda^2 + (1 - \lambda^2) \cos 2\theta \right]^{1/2}}$

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4) We have $m\ddot{\vec{r}} = -mg\hat{z} + \vec{F}_c + 2m\dot{\vec{r}} \times \vec{\omega} + m(\vec{\omega} \times \vec{r}) \times \vec{\omega}$ which we must write in polar coordinates, taking $\dot{r} = 0$

constraint force, to maintain bead on wire

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\quad\quad \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

Only keep track of $\hat{\theta}$ components, since $\vec{F}_c$ always must nullify any other acceleration → coriolis force ⊥ to hoop ⇒ ignore

$m\ddot{\vec{r}} \to mr\ddot{\theta}\,\hat{\theta} = -mg \sin\theta\, \hat{\theta} + m\omega^2 R \sin\theta \cos\theta\, \hat{\theta}$

$$\boxed{\ddot{\theta} = -\frac{g}{R} \sin\theta \left(1 - \frac{\omega^2 R}{g} \cos\theta \right)}$$