Midterm Exam Solutions

Q1) 1) all terms must have units of energy $\frac{m\ell^2}{t^2}$

$$[\lambda v^4] = [\lambda]\left(\frac{\ell}{t}\right)^4 = \frac{m\ell^2}{t^2} \Rightarrow [\lambda] = m\left(\frac{t}{\ell}\right)^2$$

$$[kx^2] = [k]\ell^2 = m\frac{\ell^2}{t^2} \Rightarrow [k] = \frac{m}{t^2}$$

$$[cv^2x^2] = [c]\left(\frac{\ell}{t}\right)^2\ell^2 = m\frac{\ell^2}{t^2} \Rightarrow [c] = \frac{m}{\ell^2}$$

2) Only one d.o.f. (x)

3) $$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -kx + 2cv^2x - \frac{d}{dt}\left[\lambda v^3 + 2cvx^2\right]$$

$$= -kx + 2cv^2x - 3\lambda v^2\dot{v} - 2c\dot{v}x^2 - 4cv^2x = -kx - 2cv^2x - 3\lambda v^2\dot{v} - 2c\dot{v}x^2 = 0$$

4) Yes, since $\mathcal{L}$ does not depend explicitly on time.

5) need to compute $E_{tot} = p\dot{q} - \mathcal{L} = (\lambda v^4 + 2cv^2x^2) - \frac{\lambda v^4}{4} + \frac{kx^2}{2} - cv^2x^2 = \frac{3}{4}\lambda v^4 + \frac{kx^2}{2} + cv^2x^2$

$$E_{tot} = \frac{3}{4}\lambda v_0^4 + \frac{kx_0^2}{2} + cv_0^2x_0^2 \qquad v_{max} \text{ @ } x = 0$$

$$v_{max} = \left(\frac{4E_0}{3\lambda}\right)^{1/4}$$

Q2) 1) naming the 3 vectors $\hat{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)$; $\hat{e}_2 = \frac{1}{\sqrt{2}}(1,-1,0)$; $\hat{e}_3 = \frac{1}{\sqrt{6}}(1,1,-2)$, we see that

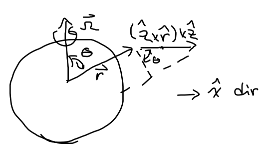
We see that in this new basis, $V = f_1(x_2) + f_2(x_3)$, so $\mathcal{L}$ does not depend explicitly on x_1

p_{x_1} is conserved, w/ $x_1 = x+y+z$ being the cyclic / ignorable coordinate.

$$p_{x_1} = p_x + p_y + p_z = m(v_x + v_y + v_z)$$

2) $p_x + p_y + p_z = m(v_x^0 + v_y^0 + v_z^0) = 6m \Rightarrow$ when $v_x = v_y = 0$, $v_z = 6$

Q3) $$m\ddot{\vec{r}} = \vec{F} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} = -mg\hat{r} + m\Omega^2 R_e \underbrace{(\hat{z}\times\hat{r})}_{\sin\theta\,(\hat{in})\times(\hat{z}) = \sin\theta\,\hat{x}}\times\hat{z} = -mg\hat{r} + m\Omega^2R_e\sin\theta\,\hat{x} = -mg\hat{r} + m\Omega^2R_e\sin\theta(\cos\theta\,\hat{south} + \sin\theta\,\hat{r})$$



$$\simeq -mg\hat{r} + m\Omega^2R_e\underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}}(\hat{south})$$

$$\boxed{\tan\alpha = \frac{\Omega^2 R_e \sin 2\theta}{2g}}$$