Lecture 09.2 - Physics 523

$\boxed{\text{Rotation of Rigid Bodies}}$ Recall that T separates into translational + rotational kinetic energy

$$T = \frac{1}{2} M \dot{\vec{R}}_{cm}^{2} + \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'^{\,2}$$

↑ position vectors rel. to $\vec{R}_{cm}$

Other quantities also separate $\vec{L} = \underbrace{\vec{R} \times M\dot{\vec{R}}}_{\text{Orbital}} + \underbrace{\sum_\alpha m_\alpha \vec{r}_\alpha' \times \dot{\vec{r}}_\alpha'}_{\text{Spin}}$

Orbital: eg Rot. of earth around sun

Spin: e.g. rotation of earth about its own axis

★ When we consider the angular momentum of a rigid body rotating about some axis, we saw that $\vec{L} \not\propto \vec{\omega}$!

Instead $\vec{L} = \hat{I} \cdot \vec{\omega}$ ($\vec{\omega}$ multiplied by <u>matrix</u>)

General Expression for angular momentum → $\vec{L} = \sum m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$

Now $\vec{\omega} \times \vec{r}_\alpha = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_x & \omega_y & \omega_z \\ x_\alpha & y_\alpha & z_\alpha \end{pmatrix} = \hat{x}(\omega_y z_\alpha - \omega_z y_\alpha) + \hat{y}(\omega_z x_\alpha - \omega_x z_\alpha) + \hat{z}(\omega_x y_\alpha - \omega_y x_\alpha)$

$$\Rightarrow \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ \omega_y z_\alpha - \omega_z y_\alpha & \omega_z x_\alpha - \omega_x z_\alpha & \omega_x y_\alpha - \omega_y x_\alpha \end{pmatrix}$$

$$= \hat{x}\left[y_\alpha(\omega_x y_\alpha - \omega_y x_\alpha) - z_\alpha(\omega_z x_\alpha - \omega_x z_\alpha)\right] + \hat{y}\left[z_\alpha(\omega_y z_\alpha - \omega_z y_\alpha) - x_\alpha(\omega_x y_\alpha - \omega_y x_\alpha)\right] + \hat{z}\left[x_\alpha(\omega_z x_\alpha - \omega_x z_\alpha) - y_\alpha(\omega_y z_\alpha - \omega_z y_\alpha)\right]$$

$$= \hat{x}\left[\omega_x(y_\alpha^2 + z_\alpha^2) - \omega_y y_\alpha x_\alpha - \omega_z z_\alpha x_\alpha\right] + \hat{y}\left[\omega_y(x_\alpha^2 + z_\alpha^2) - \omega_x x_\alpha y_\alpha - \omega_z z_\alpha y_\alpha\right] + \hat{z}\left[\omega_z(x_\alpha^2 + y_\alpha^2) - \omega_x x_\alpha z_\alpha - \omega_y y_\alpha z_\alpha\right]$$

Can write this as $m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = m_\alpha \underbrace{\begin{pmatrix} y_\alpha^2 + z_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & x_\alpha^2 + z_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & x_\alpha^2 + y_\alpha^2 \end{pmatrix}}_{\text{Symmetric matrix}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \hat{I}_\alpha \cdot \vec{\omega}$

$$\vec{L}_{tot} = \sum_\alpha \hat{I}_\alpha \cdot \vec{\omega} = \hat{I} \cdot \vec{\omega}$$

$I = I^T$ (transpose of I → $(I^T)_{ij} = (I)_{ji}$)

Aside get used to matrix algebra in classical mechanics → extremely powerful

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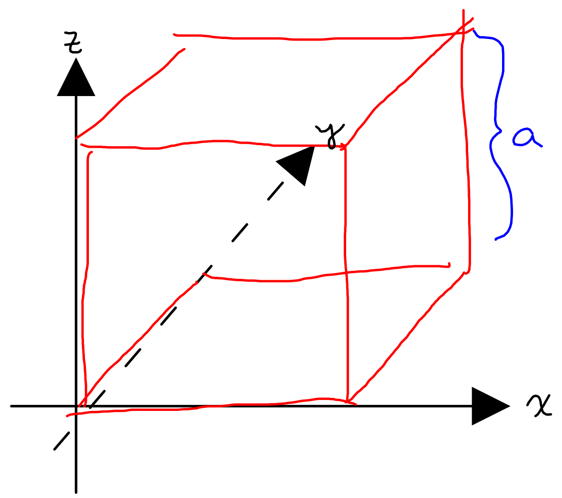
Example: Inertia tensor of solid cube w/ sides of length a, density ρ, rotating about axes passing through corner.



$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) \rightarrow \int \rho (y^2+z^2)\, dV = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho (y^2+z^2)$$

$$= \frac{2}{3} a^5 \rho = \frac{2}{3} M a^2$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = -\int_0^a dx \int_0^a dy \int_0^a dz\, \rho\, xy = -\frac{1}{4} \rho a^5 = -\frac{1}{4} M a^2$$

All other terms are similar: $I_{xx} = I_{yy} = I_{zz}$ $\quad I_{xy} = I_{xz} = \cdots$

$$\hat{I} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Angular momentum of this cube $\rightarrow$ $\vec{L} = \hat{I}\, \vec{\omega}$ $\quad$ say $\vec{\omega}$ is along $\hat{x}$ axis $\Rightarrow$ $\vec{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$

$$\vec{L} = \hat{I} \cdot \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 8\omega \\ -3\omega \\ -3\omega \end{pmatrix}$$

Again, $\vec{L}$ & $\vec{\omega}$ are not $\parallel$ vectors!

Let's try another axis of rotation: $\vec{\omega} = \frac{\omega}{\sqrt{3}} (1,1,1)$ points from corner at $\odot$ to far corner

$$\vec{L} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{3}} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{3}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \frac{Ma^2}{6} \left[\frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] = \frac{Ma^2}{6} \vec{\omega}$$

for this special $\vec{\omega}$, $\vec{L}$ is $\parallel$ to $\vec{\omega}$!

What about rotation about cm? (Middle of cube naturally)

$$I_{xx} = \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz\, \rho\,(y^2+z^2) = \frac{\rho a^5}{6} = \frac{1}{6} M a^2$$

similar for rest

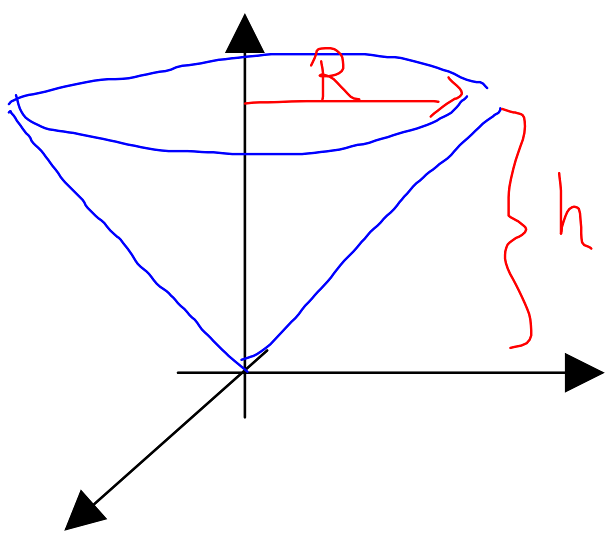
$$I_{xy} = -\int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz\, \rho\, xy = 0$$

$$\hat{I} = \frac{Ma^2}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proportional to identity!

$\vec{L}$ always $\parallel$ to $\vec{\omega}$!

Can also show $\hat{I}$ for cone is



$$\hat{I} = \frac{3}{20} M \begin{pmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}$$

Diagonal, but not $\propto \mathbb{1}$ matrix

Clearly, some choices of pivot points + coordinate systems are special

"Principle Axes" of inertia

$\boxed{\text{Principle Axes}}$ An axis is a "principle axis" if rotation about that axis gives $\vec{L} \parallel \vec{\omega}$

If $\hat{I}$ is diagonal for a choice of x-y-z axes, then $\hat{x}, \hat{y}, + \hat{z}$ directions are principle axes

Explicitly $\hat{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$

$$\hat{I}\,\vec{\omega}_x = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = I_1 \vec{\omega}_x \Rightarrow \vec{L} \parallel \text{ to } \vec{\omega}$$

similar for $\vec{\omega}_y$, $\vec{\omega}_z$

I_1, I_2, I_3 are "principle moments" corresponding to the principle axes

Axes of symmetry are principle axes, but symm. not a necessary condition

→ any rigid body has 3 principle axes w/ 3 associated principle moments.

Also, principle axes form orthogonal set (always ⊥ to each other)

<u>Note</u> once you have the inertia tensor, you can calculate the kinetic energy of rotation

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}^T\cdot\hat{I}\cdot\vec{\omega} = \frac{1}{2}(\omega_x, \omega_y, \omega_z)\,\hat{I}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} \overset{\text{principle axes, } \hat{I} \text{ diagonal}}{=} \frac{1}{2}\sum_i \omega_i^2 \underbrace{I_i}_{\text{principle moments}}$$

Say you don't know P.A.'s, but you know $\hat{I}$ in some coordinates → how do you find P.A.'s?

eigenvalue / eigenvector problem! → Knowing how to deal with these is crucial in physics

Our goal is this → Find solutions to $\hat{I}\vec{\omega} = \lambda\vec{\omega}$ (find special $\vec{\omega}$'s (P.A.'s) & associated λ's (P. moments))

Rewrite $(\hat{I} - \lambda\mathbb{1})\vec{\omega} = 0$ If $\vec{\omega}$ is non-trivial, require $\det(\hat{I} - \lambda\mathbb{1}) = 0$ Cubic in $\lambda \Rightarrow$ 3 solutions

<u>Example</u> Cube w/ corner at origin $\hat{I} = \frac{Ma^2}{12}\begin{pmatrix}8 & -3 & -3\\ -3 & 8 & -3\\ -3 & -3 & 8\end{pmatrix}$ $\frac{Ma^2}{12}\left[\begin{pmatrix}8 & -3 & -3\\ -3 & 8 & -3\\ -3 & -3 & 8\end{pmatrix} - \tilde{\lambda}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] = \frac{Ma^2}{12}\begin{pmatrix}8-\tilde{\lambda} & -3 & -3\\ -3 & 8-\tilde{\lambda} & -3\\ -3 & -3 & 8-\tilde{\lambda}\end{pmatrix}$ ($\tilde{\lambda} = \lambda \cdot \frac{12}{Ma^2}$) take det: $(8-\tilde{\lambda})^3 + 54 - 27(8-\tilde{\lambda}) = 0$

Solutions $\tilde{\lambda} = 2, 11$ (2 degenerate eigenvalues)

<u>Eigenvectors</u> 1) $(\hat{I} - 2\mathbb{1})\vec{\omega} = \begin{pmatrix}6 & -3 & -3\\ -3 & 6 & -3\\ -3 & -3 & 6\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = 0$ 3 lin. eq.s → $\vec{\omega} = \frac{\omega}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$ (the one we saw earlier)

2) $(\hat{I} - 11\mathbb{1})\vec{\omega} = \begin{pmatrix}-3 & -3 & -3\\ -3 & -3 & -3\\ -3 & -3 & -3\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$ only one eq.! $\omega_x + \omega_y + \omega_z = 0$

(orthogonal to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ vector)