Lecture 09.1 - Physics 523

$\boxed{\text{Rigid Body Rotation}}$ Recall that a rigid 3D body has at most 6 d.o.f $\vec{X}_{cm}$ + 3 angles

Also recall we can recast dynamics in terms of (often conserved) $\vec{P}_{cm}$ (often conserved as $\vec{X}_{cm}$ is often cyclic) and rotational dof.

↳ these will be our focus.

Rigid Body: N dof, constrained to move in unison

$$\vec{R}_{cm} = \underbrace{\frac{1}{M}}_{\text{total mass}} \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \rightarrow \vec{R}_{cm} = \frac{1}{M}\int_V \rho \vec{r}\, dV$$

↳ over volume (for cont. mass distribution)

Recall: the CM dynamics obey's Newton's laws like individual coordinates

$$\vec{P} = \sum_\alpha \vec{P}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}_{cm}$$

$$\boxed{\vec{F}_{ext} = M\ddot{\vec{R}}_{cm}}$$

Rigid bodies behave in much the same way as point particles

$\boxed{\text{Angular Momentum}}$ Just as eq's for linear motion separate CM from rotational, the same happens for angular momentum

Write position's w.r.t. CM: $\vec{r}_\alpha = \vec{R}_{cm} + \vec{r}_\alpha{}'$

vector from CM to position of m_α

$$\vec{L}_{tot} = \sum_\alpha \vec{r}_\alpha \times (m_\alpha \dot{\vec{r}}_\alpha) = \sum m_\alpha (\vec{R} + \vec{r}_\alpha{}') \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}')$$

$$= \sum_\alpha \left(m_\alpha \vec{R} \times \dot{\vec{R}} + \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'\right) + \vec{R} \times \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}' + \left(\sum_\alpha m_\alpha \vec{r}_\alpha{}'\right) \times \dot{\vec{R}}$$

$\vec{R}_{cm}' + \vec{R}_{cm}' = 0$

$$\boxed{\vec{L}_{tot} = \underbrace{\vec{R} \times (M\dot{\vec{R}})}_{\vec{L}_{cm}} + \underbrace{\sum_\alpha \vec{r}_\alpha{}' \times (m_\alpha \dot{\vec{r}}_\alpha{}')}_{\vec{L}_{rot.}}}$$

★ So angular momentum separates into $\vec{L}$ of CM + $\vec{L}$ of rotation of body about axes passing through CM.

Compare to study of orbitals in atoms, in which $\underbrace{\vec{L}_{tot} = \vec{L}_{orb.}}_{\text{orbital } \vec{L}} + \underbrace{\vec{L}_{spin}}_{\vec{L} \text{ due to spin of electron}}$ (also true of planetary rotation in solar system)

★ Time rate of change: $\dot{\vec{L}}_{orb} = \frac{d}{dt}\left[\vec{R}\times\vec{P}\right] = \cancel{\dot{\vec{R}}\times\vec{P}}^{\,0} + \vec{R}\times\dot{\vec{P}} = \vec{R}\times\vec{F}_{ext}$

for central forces, $\vec{F}_{ext} \parallel$ to $\vec{R}_{cm} \Rightarrow \dot{\vec{L}}_{orb} = 0$ Conserved!

$$\dot{\vec{L}}_{spin} = \dot{\vec{L}}_{tot} - \dot{\vec{L}}_{orb} = \vec{\tau}_{ext} - \vec{R}\times\vec{F}_{ext} = \sum_\alpha \vec{r}_\alpha{}'\times\vec{F}_\alpha^{ext} = \vec{\tau}_{cm}$$ (torque about CM) Quite sensible!

$$\sum_\alpha \vec{r}_\alpha\times\vec{F}_\alpha^{ext} = \sum_\alpha(\vec{R}+\vec{r}_\alpha{}')\times\vec{F}_\alpha^{ext} = \vec{R}\times\vec{F}_{ext} + \sum_\alpha \vec{r}_\alpha{}'\times\vec{F}_\alpha^{ext}$$

cancels against other term

$\boxed{\text{Kinetic Energy}}$ Another simplification for KE:

$$T = \sum_\alpha \tfrac{1}{2}m_\alpha\dot{\vec{r}}_\alpha^{\,2} = \sum_\alpha \tfrac{1}{2}m_\alpha(\dot{\vec{R}}+\dot{\vec{r}}_\alpha{}')^2 = \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\dot{\vec{R}}_\alpha\cdot\cancelto{0}{(\textstyle\sum m_\alpha\dot{\vec{r}}_\alpha{}')} + \tfrac{1}{2}\sum_\alpha m_\alpha\dot{\vec{r}}_\alpha{}'^2$$

$$T = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{T_{cm}} + \underbrace{\tfrac{1}{2}\sum_\alpha m_\alpha\dot{\vec{r}}_\alpha{}'^2}_{\sum_\alpha T_{rel}}$$ (energy of rotation

e.g. KE of rigid body is translational + rotational

$\boxed{\text{Potential Energy}}$ Recall that, early on, we showed $U = \underbrace{U^{ext}}_{\text{potential E due to ext. conservative forces}} + \underbrace{U^{int}}_{\sum_\alpha U_{\alpha\beta}(\vec{r}_{\alpha\beta})} = U^{ext}$ (may depend on all 6 dof for rigid body)

never change in rigid body $\Rightarrow U^{int} = $ const. (Set to zero!)

Rotation about fixed axis We have seen that CM motion + rotational motion separate in all respects $\Rightarrow$ focus on rotation about CM

Call rotation axis the $\hat{z}$-axis $\vec{\omega}$ (e.g. spinning figure skater) $\quad \vec{L} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$ and $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha$

now $\vec{\omega}$ is along z-axis $\Rightarrow$ $\vec{\omega} \times \vec{r}_\alpha = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{pmatrix} = \omega x_\alpha \hat{y} - \omega y_\alpha \hat{x}$

$$\vec{\ell}_\alpha = m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = m_\alpha \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{pmatrix} = m_\alpha \omega \Big[\underbrace{\hat{x}(-z_\alpha x_\alpha) + \hat{y}(-z_\alpha y_\alpha)}_{\text{not generally } \emptyset \text{ !!!}} + \hat{z}(x_\alpha^2 + y_\alpha^2) \Big]$$

squared dist. to z-axis

$\vec{L} = I\vec{\omega}$ NOT TRUE $\qquad L_x = -\omega \sum_\alpha m_\alpha x_\alpha z_\alpha \neq 0$

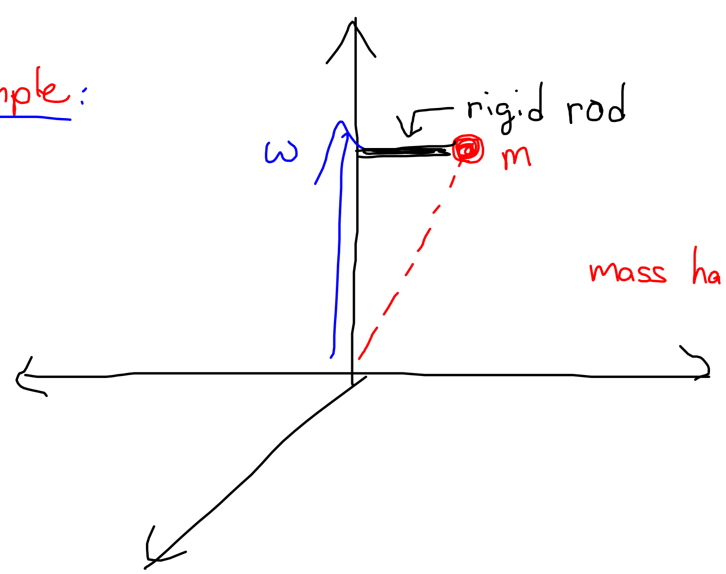
$L_y = -\omega \sum_\alpha m_\alpha y_\alpha z_\alpha \neq 0$

Correct prescription: $\vec{L} = \hat{I} \cdot \vec{\omega}$

moment of inertia matrix

(principle axes)

Example:



If mass were in x-y plane, $L_x = L_y = 0$, but $L_{x,y}$ do not vanish!

mass has comp. of $\vec{L}$ rel. to origin

$$\vec{L}_{tot} = I_{xz}\omega \hat{x} + I_{yz}\omega \hat{y} + I_{zz}\omega \hat{z}$$

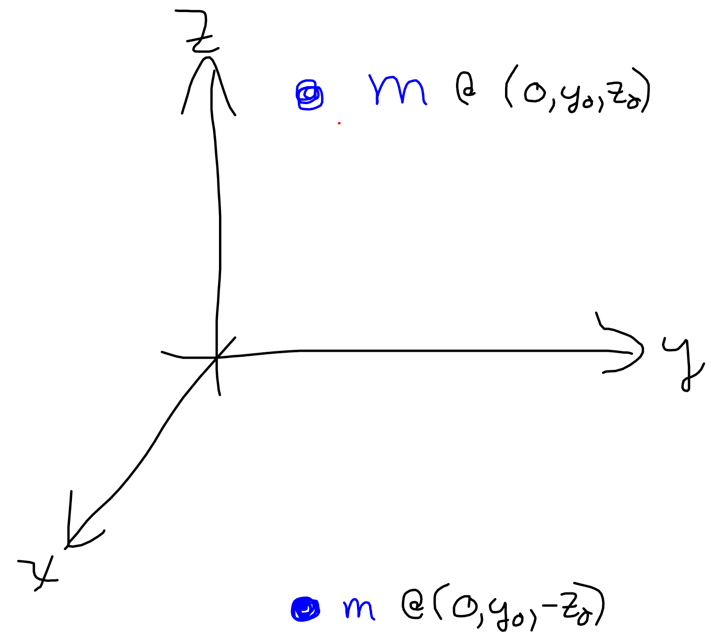
$$\boxed{\begin{aligned} I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha \end{aligned}}$$

"Products of inertia"

$$\boxed{I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2}$$

$x_\alpha^2 + y_\alpha^2$

$\boxed{\text{Examples}}$ Calculate I_{xz}, I_{yz}, I_{zz} for a)

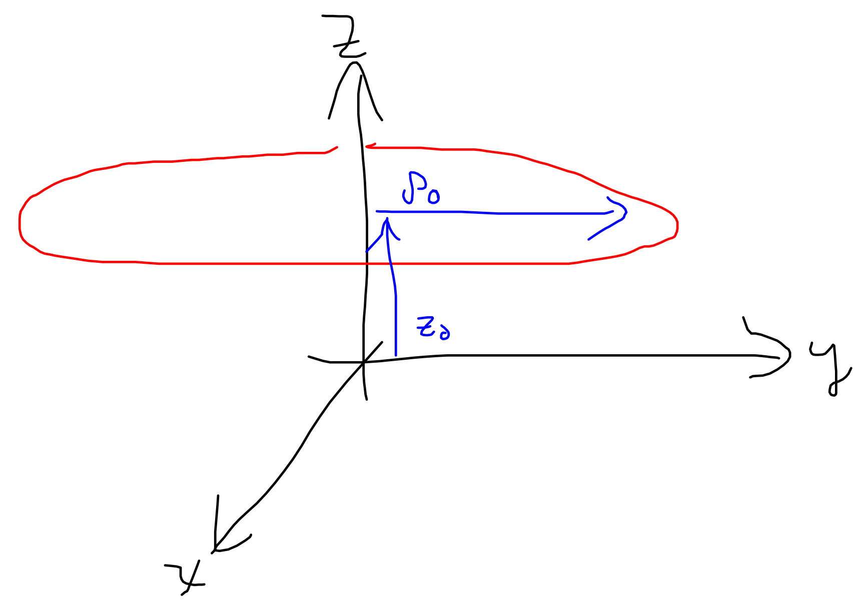


$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = 0$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = 0$$

$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = 2 m y_0^2$$

b)



$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = 0 \quad \text{by sym.}$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = 0 \quad \text{by same}$$

$$I_{zz} = +\sum_\alpha m_\alpha \rho_\alpha^2 = M r_0^2$$

Moral: exploit symmetries to simplify calculation of products / moments of inertia!