Lecture 07.2 - Physics 523

Rotating Reference frames Usually ref. frame is some rotating rigid body (e.g. earth, or spaceship w/artificial gravity...)

Euler's Theorem If point P in a body is fixed, then most general motion is rotation about some axis through P.
(fairly intuitive)

Rate of rotation: Angular velocity ω $\rightarrow$ can give it a dir., due to Euler's thm. $\vec{\omega} = \omega \hat{u}$ (use RHR to specify sign)
↑ lies along axis of rotation through P.

Generalization of $v = \omega r$: $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$ or $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$ in general, for fixed vector $\vec{a}$ in frame S_1
$\frac{d\vec{a}}{dt} = \vec{\omega} \times \vec{a}$ in frame S_0

Addition of angular velocities: 3 frames, S_0, S_1, + S_2 S_1 has $\vec{\omega}_{10}$ relative to S_0 S_2 has $\vec{\omega}_{21}$ rel. to S_1

What is $\vec{\omega}_{20}$? We know translational velocities add

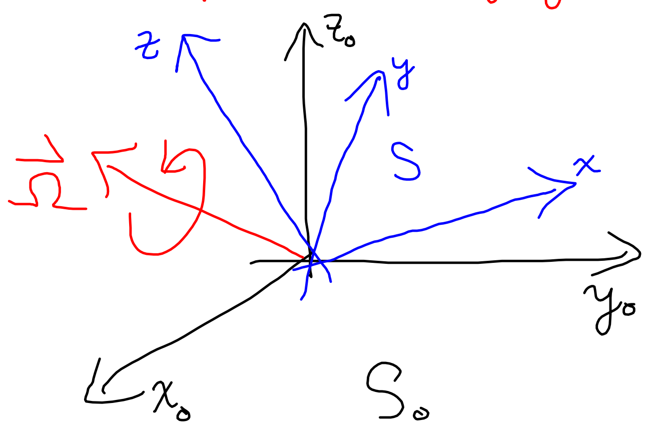
$\vec{V}_{20} = \vec{V}_{10} + \vec{V}_{21}$ $\vec{\omega}_{20} \times \vec{r} = \vec{\omega}_{10} \times \vec{r} + \vec{\omega}_{21} \times \vec{r} = (\vec{\omega}_{10} + \vec{\omega}_{21}) \times \vec{r}$

$$\boxed{\vec{\omega}_{20} = \vec{\omega}_{10} + \vec{\omega}_{21}}$$

We want eom in frame S, rotating w/ ang. freq. $\vec{\Omega}$ relative to another frame S_0

eg. frame S rotating along w/ earth $\rightarrow$ $|\vec{\Omega}_{earth}| = 7.3\times10^{-5}$ rad/s

For the purpose of our ongoing discussion, ignore other rotations (e.g. earth-moon system)



Consider vector $\vec{a}$ (any physical quantity)

Want to relate $\left.\frac{d\vec{a}}{dt}\right|_{S_0}$ to $\left.\frac{d\vec{a}}{dt}\right|_{S}$

unit vectors fixed in S

$$\vec{a}\,\Big|_S = \sum_i a_i \hat{e}_i \qquad \left.\frac{d\vec{a}}{dt}\right|_S = \sum_i \frac{da_i}{dt}\hat{e}_i \qquad \left(\left.\frac{d\hat{e}_i}{dt}\right|_S = 0\right)$$

But $\hat{e}_i$ not fixed in S_0 frame $\left.\frac{d\hat{e}_i}{dt}\right|_{S_0} = \vec{\Omega}\times\hat{e}_i$ from earlier discussion

$$\left.\frac{d\vec{a}}{dt}\right|_{S_0} = \sum_i \frac{da_i}{dt}\hat{e}_i + a_i\frac{d\hat{e}_i}{dt} = \underbrace{\sum_i \frac{da_i}{dt}\hat{e}_i}_{\dot{a}|_S} + \underbrace{\sum_i a_i\,\vec{\Omega}\times\hat{e}_i}_{=\vec{\Omega}\times\left(\sum_i a_i\hat{e}_i\right) = \vec{\Omega}\times\vec{a}} = \left.\frac{d\vec{a}}{dt}\right|_S + \vec{\Omega}\times\vec{a}$$

$$\boxed{\left.\frac{d\vec{a}}{dt}\right|_{S_0} = \left.\frac{d\vec{a}}{dt}\right|_S + \vec{\Omega}\times\vec{a}}$$

We can use this relation to express Newton's EoM's in arbitrary rotating frame

Newton's 2nd

$\vec{F} = m\ddot{\vec{r}}\,\big|_{S_0}$ S_0 is inertial frame

↳ need this vector in arb. rotating frame S, $\vec{\Omega}$ rel. to S_0

$$\vec{v}\,\big|_{S_0} = \frac{d\vec{r}}{dt}\Big|_{S_0} = \frac{d\vec{r}}{dt}\Big|_{S} + \vec{\Omega}\times\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2}\Big|_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left[\frac{d\vec{r}}{dt}\Big|_S + \vec{\Omega}\times\vec{r}\right] = \left(\frac{d}{dt}\right)_{S}\left[\frac{d\vec{r}}{dt}\Big|_S + \vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times\left[\frac{d\vec{r}}{dt}\Big|_S + \vec{\Omega}\times\vec{r}\right]$$

$$= \frac{d^2\vec{r}}{dt^2}\Big|_S + 2\vec{\Omega}\times\frac{d\vec{r}}{dt}\Big|_S + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

S_0 $$\vec{F} = m\ddot{\vec{r}}\,\big|_{S_0} = m\ddot{\vec{r}}\,\big|_S + 2m\,\vec{\Omega}\times\dot{\vec{r}}\,\big|_S + m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\boxed{m\ddot{\vec{r}}\,\big|_S = \vec{F} + \overbrace{\underbrace{2m\,\dot{\vec{r}}\times\vec{\Omega}\,\big|_S}_{\text{coreolis force}} + \underbrace{m\,(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}_{\text{centrifugal force}}}^{\text{just } -m\vec{A} \text{ for this frame}}}$$

↑ this was vanishing in our examples on tuesday

Centrifugal Force

If relative velocity of 2 frames vanishes, as in our examples so far, $\vec{F}_{cor} = 0$

→ small when object at rest in rotating frame, or $\dot{\vec{r}}$ is small

$$\frac{F_{cor}}{F_{cf}} \sim \frac{v}{R\Omega} = \frac{v}{V}$$

↑ for earth, ~ 10^3 mi/hr near equator

$$\vec{F}_{cf} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$