Lecture 06.2 - Physics 523

Solving the Kepler Problem

Solving for $r(t)$ is still tricky

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

sub. $r = 1/u$, $\quad \frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\ell}{\mu r^2}\frac{d}{d\theta} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$

So $\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\theta}$

$\ddot{r} = -\frac{\ell u^2}{\mu}\frac{d}{d\theta}\left(\frac{\ell}{\mu}\frac{du}{d\theta}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\theta^2}$

New eq. $\quad -\frac{\ell^2 u^2}{\mu}u'' = F(u) + \frac{\ell^2 u^3}{\mu} \Rightarrow u'' = -u - \frac{\mu}{\ell^2 u^2}F(u) \overset{F=-\frac{\gamma}{r^2}}{=} -u + \frac{\mu}{\ell^2}\gamma$

Linear, but inhomogenous: $\quad u'' = -u + \frac{\mu}{\ell^2}\gamma \quad$ sub. $w = u - \frac{\mu}{\ell^2}\gamma \Rightarrow w'' = -w \quad w = A\cos(\phi - \delta)$

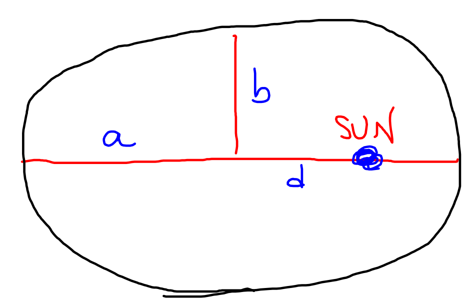
$$u(\phi) = \frac{\mu}{\ell^2}\gamma\left(1 + \varepsilon\cos(\phi+\delta)\right) \qquad r = \frac{1}{u} = \frac{r_0}{1+\varepsilon\cos(\theta-\delta)} \qquad r_0 = \frac{\ell^2}{\mu\gamma} = \frac{\ell^2}{\mu}\frac{1}{Gm_1m_2}$$

$\varepsilon < 1$ Orbit is bounded $\quad r_{min} = \frac{r_0}{1+\varepsilon} \quad r_{max} = \frac{r_0}{1-\varepsilon}$ $\quad$ Motion is ellipse, w/ eccentricity ε

$\varepsilon \geq 1$ Orbit unbounded $\quad r \to \infty$ for $\cos(\theta - \delta) = -1/\varepsilon$

motion is $\varepsilon = 1 \to$ parabola

$\varepsilon > 1 \to$ hyperbola



$a = \frac{r_0}{1-\varepsilon^2} \qquad b = \frac{c}{\sqrt{1-\varepsilon^2}}$

$d = a\varepsilon$

$\varepsilon = 0 \to$ circle

In terms of ℓ, E

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left[\frac{\ell^2}{\mu r_{min}} - 2\gamma\right]$$

$$r_{min} = \frac{r_0}{1+\varepsilon} = \frac{\ell^2/\mu\gamma}{1+\varepsilon}$$

$$E = \frac{\mu\gamma(1+\varepsilon)}{2\ell^2}\left[\gamma(1+\varepsilon) - 2\gamma\right] = \frac{\mu\gamma^2}{2\ell^2}(\varepsilon+1)(\varepsilon-1) = \frac{\mu\gamma^2}{2\ell^2}(\varepsilon^2-1)$$

get ε in terms of E + ℓ } DONE
r_0 in terms of ℓ

Rocket Boosters Changing orbit: $(E_1, \ell_1) \to (E_2, \ell_2)$ Imagine a tangential impulse at r_{min} (perigee)

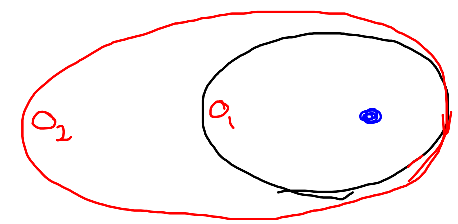
$$\frac{r_0^{(1)}}{1+\varepsilon_1} = \frac{r_0^{(2)}}{1+\varepsilon_2}$$ (perigees agree)

Say $\ell_2 = \lambda \ell_1$ ← thrust factor $(v_2 = \lambda v_1)$

$$r_0^{(1)} = \frac{\ell_1^2}{\mu\gamma} \qquad r_0^{(2)} = \frac{\ell_2^2}{\mu\gamma} = \lambda^2 r_0^{(1)}$$

$$\frac{r_0^{(1)}}{1+\varepsilon_1} = \frac{\lambda^2 r_0^{(1)}}{1+\varepsilon_2}$$

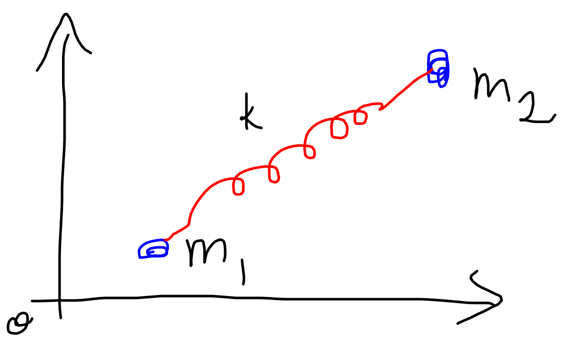
$\lambda > 1, \ \varepsilon_2 > \varepsilon_1$
$\lambda < 1, \ \varepsilon_2 < \varepsilon_1$ } modify eccentricity



$\lambda > 1$ in this case

Parting Words In eliminating the cyclic/ignorable coordinate ℓ, we effectively went to a rotating frame
Non-inertial → next time we will consider general treatment of non-inertial ref. frames.

Central Forces – 2 body problem



1) Write $\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - \frac{1}{2}k(|\vec{r}_1-\vec{r}_2|-r_0)^2$

$$= \frac{1}{2}M\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}\mu(\dot{\vec{r}}_1-\dot{\vec{r}}_2)^2 - \frac{1}{2}k(|\vec{r}_1-\vec{r}_2|-r_0)^2$$

$$= \frac{1}{2}M\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - \frac{1}{2}k(|\vec{r}|-r_0)^2$$

↳ ignorable gen. momentum $\vec{P}_{tot}$ → goto frame w/ $\vec{P}_{tot}=0$

$$\longrightarrow \frac{1}{2}\mu(\dot{r}^2+r^2\dot{\theta}^2) - \frac{1}{2}k(r-r_0)^2$$

↑ ignorable gen. momentum $\ell = \mu r^2\dot{\theta}$ conserved

$$\longrightarrow \frac{1}{2}\mu\left(\dot{r}^2+\frac{\ell^2}{\mu^2r^2}\right) - \frac{1}{2}k(r-r_0)^2$$

$$\longrightarrow \frac{1}{2}\mu\dot{r}^2 - \left[\frac{1}{2}k(r-r_0)^2 - \frac{\ell^2}{2\mu r^2}\right]$$ Only 1 coordinate left!

EoM $\frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = -k(r-r_0) - \frac{\ell^2}{\mu r^3} - \mu\ddot{r} = 0$ $\boxed{\mu\ddot{r} = -k(r-r_0) - \frac{\ell^2}{\mu r^3}}$

Non-inertial Frames

Often want to do physics in N.I. frames (e.g. surface of the earth)

Consider 2 reference frames $\underbrace{\text{One inertial}}_{S_0}$, $\underbrace{\text{other non-inertial}}_{S}$ S accelerating w/ accel. $\vec{A}(t)$ (eg. roller-coaster) } eg. flipping coin

A trajectory in S_0 satisfies $m\ddot{\vec{r}}_0 = \vec{F}$ in S: $\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$ ($\vec{v}$ in S) $\ddot{\vec{r}} = \ddot{\vec{r}}_0 - \dot{\vec{A}}$ $m\ddot{\vec{r}} = m\ddot{\vec{r}}_0 - m\vec{A} = \vec{F} - m\vec{A}$

supplement Newton's Laws w/ "inertial force"

Inertial force → Coffee in mug in car Sloshes to back of mug → like relative force pushing back on liquid

Tidal Forces

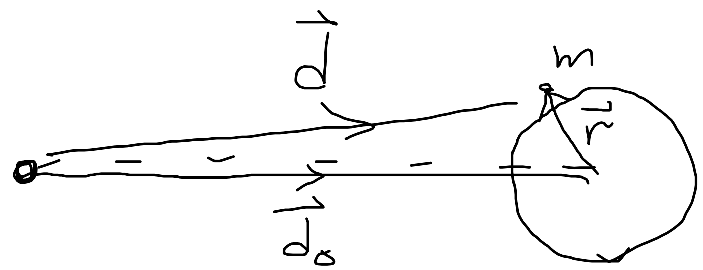
The earth's oceans respond to the relative orientations of the earth + moon

Forces on mass m on surface of earth

1) $-mg\hat{r}$ earths center is origin

2) $F_{moon} = -\dfrac{GM_{\circ}m}{d^2}\hat{d}$

3) Other local forces, $\vec{F}_{ng}$



Earth has centripetal accel, $\vec{A} = -\dfrac{GM_{\circ}}{d_{\circ}^2}\hat{d}_{\circ}$

Using non-inertial frame mod. of Newton's 2nd $m\ddot{\vec{r}} = \vec{F} - m\vec{A}$: $m\ddot{\vec{r}} = -mg\hat{r} - GM_{\circ}m\dfrac{\hat{d}}{d^2} + \vec{F}_{ng} + GM_{\circ}m\dfrac{\hat{d}_{\circ}}{d_{\circ}^2}$

$$\boxed{m\ddot{\vec{r}} = -mg\hat{r} + \vec{F}_{ng} + \vec{F}_{tidal}}$$

$$F_{tidal} = -GM_{\circ}m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_{\circ}}{d_{\circ}^2}\right)$$