Lecture 06.1 - Physics 523

Conservation of Energy

$\mathcal{L} = T - U \rightarrow$ NOT conserved

$$\frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i; t) = \underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{= \frac{d}{dt} p_i} \dot{q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p} \ddot{q} + \frac{\partial \mathcal{L}}{\partial t}$$

Generally

$$\frac{d\mathcal{L}}{dt} = \sum_i \dot{p}_i \dot{q}_i + \sum p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} = \sum_i \frac{d}{dt}(p_i \dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}$$

If $\mathcal{L}$ has no explicit time dependence, $\frac{\partial \mathcal{L}}{\partial t} = 0$, & $\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\sum_i (p_i \dot{q}_i)$

Define $\mathcal{H} = \sum_i \overbrace{p_i \dot{q}_i}^{2T \text{ in cartesian}} - \mathcal{L}$

$2T - (T-U) = T+U = E_{tot}$

$\frac{d}{dt}\mathcal{H} = 0$

$\mathcal{H}$ is the "Hamiltonian"

This quantity forms the backbone of Hamiltonian mechanics (and eventually quantum mech!

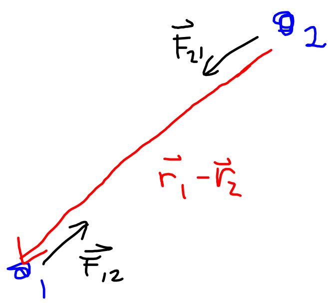
2 Body Central Force problems

2 particles w/ masses m_1 and m_2

Central conservative forces

along $\vec{r} = \vec{r}_1 - \vec{r}_2$

path indep. W, function of $\vec{r}$ only } define $U(\vec{r}_1, \vec{r}_2)$



Examples 1) Gravity $\rightarrow$ $U = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$

Coordinate choice $\rightarrow$ only func. of $\vec{r}_1 - \vec{r}_2$

2) Electric force (Hydrogen) $U = -\dfrac{ke^2}{|\vec{r}_1 - \vec{r}_2|}$

conservative, $U(\vec{r}_1, \vec{r}_2) = U(r = |\vec{r}_1 - \vec{r}_2|)$

The Lagrangian

$$\mathcal{L} = T - U = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$$

Recall HW#2:

$$M^2 \vec{R}_{cm}^{\,2} = M \sum r_i^2 m_i - \tfrac{1}{2} \sum_{i,j} m_i m_j (\vec{r}_i - \vec{r}_j)^2$$

also $M^2 \dot{\vec{R}}_{cm}^{\,2} = M \sum \dot{\vec{r}}_i^{\,2} m_i^2 - \tfrac{1}{2} \sum_{i,j} m_i m_j \dot{\vec{r}}_{ij}^{\,2}$

$$\Rightarrow T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \tfrac{1}{2} M \dot{\vec{R}}_{cm}^{\,2} + \tfrac{1}{4} \sum_{i,j} \frac{m_i m_j}{M} \dot{\vec{r}}_{ij}^{\,2}$$

$\rightarrow$ KE written in terms of CM + rel. position!

$$= \tfrac{1}{2}\Big(M \dot{\vec{R}}_{cm}^{\,2} + \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{\text{"Reduced mass" } \mu \,<\, m_1, m_2} \dot{\vec{r}}^{\,2} \Big)$$

This is very useful $\rightarrow$ eliminate overall motion of system, frame w/ $\underbrace{\dot{\vec{R}}_{cm} = 0}$ ignorable coordinate $\vec{P}_{tot}$ conserved

"center of mass" frame O at $\vec{R}_{cm}$

$$\mathcal{L}_{CM} = \tfrac{1}{2} \mu \dot{\vec{r}}^{\,2} - U(r) \Rightarrow \text{single body problem}$$

for motion in plane, $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ ($\vec{r}$ & $\dot{\vec{r}}$ span at most a plane $\rightarrow$ motion 1D or 2D)

$$= \tfrac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

θ is another ignorable coord! $\Rightarrow \vec{\ell}$ conserved

What is $\vec{\ell}$? $\vec{\ell} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$

In cm frame $\vec{r}_1 = \dfrac{m_2}{M} \vec{r}$; $\vec{r}_2 = -\dfrac{m_1}{M} \vec{r}$ $\quad = \dfrac{m_1 m_2}{M} \vec{r} \times \dot{\vec{r}} = \vec{r} \times (\mu \dot{\vec{r}})$

Again, like $\vec{\ell}$ of 1 part. w/ mass μ

Note $\rightarrow$ we just proved motion is in plane $\rightarrow$ constant $\vec{r} \times \dot{\vec{r}}$ $\rightarrow$ dir $\perp$ to plane

EoM $\quad \frac{\partial \mathcal{L}}{\partial \dot\theta} = \mu r^2 \dot\theta = \ell = \text{const}$ (ignorable!)

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} = \frac{\mu(r\dot\theta)^2}{r} - \frac{\partial U}{\partial r} - \mu \ddot r = 0$$

↑ centripetal accel. $\frac{\ell^2}{\mu r^3} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2}\right)$ "U_{cent}"

$$\boxed{\frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r} - \mu \ddot r = 0}$$

Single part. w/ force $\frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$

Constant r? No problem! $f(r,\theta) = r = R$

$$\mathcal{L}' = \mathcal{L} + \lambda(r - R)$$

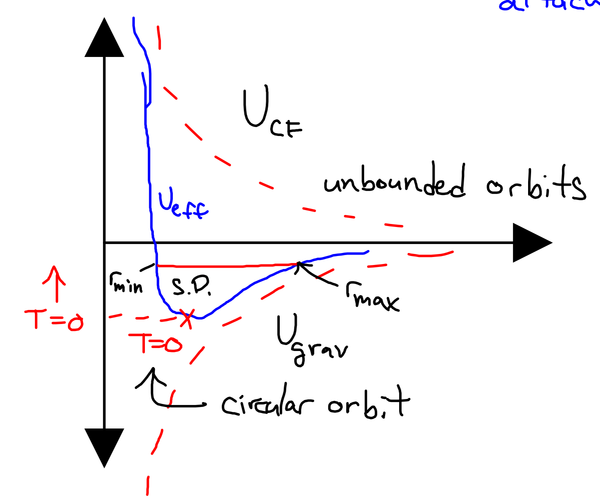
$$\mu R^2 \dot\theta = \ell = \text{const}$$

$$-\frac{\partial U}{\partial r} + \lambda + \frac{\mu(r\dot\theta)^2}{r} = 0$$

↑ force of constraint $\quad$ eg. charged pith balls attached by string

Effective potential $\quad U_{eff} = U(r) + \frac{\ell^2}{2\mu r^3}$

e.g. Gravity → $U_{eff} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$



Orbits all characterized by ℓ and E

$$\rightarrow E_{tot} = \tfrac{1}{2}\mu \dot r^2 + U_{eff}(r)$$

$$= \tfrac{1}{2}\mu \dot r^2 + \frac{(\mu r^2\dot\theta)^2}{2\mu r^2} + U_g(r)$$

$$= \tfrac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) + U_g(r) \rightarrow \text{original } T+U!$$

at r_{min}/r_{max}; $E = U_{eff}(r_{min}) = U_{eff}(r_{max})$

Summary so far: We used what we learned about Lagrangian formalism to eliminate ignorable / cyclic coordinates

→ Keplerian orbits reduced to 1D problem for $r(t)$

Works for any central force: $\quad U_{eff} = \frac{\ell^2}{2\mu r^2} + U(r)$

↑ conservative

$\boxed{\text{Solving the Kepler Problem}}$

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

Solving for $r(t)$ is still tricky

sub. $r = 1/u$, $\quad \frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\ell}{\mu r^2}\frac{d}{d\theta} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$

So $\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\theta}$

$$\ddot{r} = -\frac{\ell u^2}{\mu}\frac{d}{d\theta}\left(\frac{\ell}{\mu}\frac{du}{d\theta}\right) = -\frac{\ell^2u^2}{\mu^2}\frac{d^2u}{d\theta^2}$$

New eq. $\quad -\frac{\ell^2u^2}{\mu}u'' = F(u) + \frac{\ell^2u^3}{\mu} \Rightarrow u'' = -u - \frac{\mu}{\ell^2u^2}F(u) \overset{F=-\frac{\gamma}{r^2}}{=} -u + \frac{\mu}{\ell^2}\gamma$

Linear, but inhomogeneous: $\quad u'' = -u + \frac{\mu}{\ell^2}\gamma \qquad$ sub. $w = u - \frac{\mu}{\ell^2}\gamma \to w'' = -w \qquad w = A\cos(\phi-\delta)$

$$u(\phi) = \frac{\mu}{\ell^2}\gamma\left(1+\varepsilon\cos(\phi+\delta)\right) \qquad r = \frac{1}{u} = \frac{r_0}{1+\varepsilon\cos(\theta-\delta)} \qquad r_0 = \frac{\ell^2}{\mu\gamma} = \frac{\ell^2}{\mu}\frac{1}{Gm_1m_2}$$

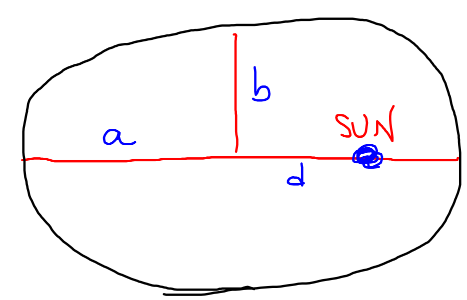
$\varepsilon < 1$ Orbit is bounded $\quad r_{min} = \frac{r_0}{1+\varepsilon} \quad r_{max} = \frac{r_0}{1-\varepsilon}$

Motion is ellipse, w/ eccentricity ε

$\varepsilon \geq 1$ Orbit unbounded $\quad r \to \infty$ for $\cos(\theta-\delta) = -1/\varepsilon$

motion is $\varepsilon = 1 \to$ parabola

$\varepsilon > 1 \to$ hyperbola



$a = \frac{r_0}{1-\varepsilon^2} \qquad b = \frac{c}{\sqrt{1-\varepsilon^2}}$

$d = a\varepsilon$

$\varepsilon = 0 \to$ circle

$\boxed{\text{In terms of } \ell, E}$

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2r_{min}}\left[\frac{\ell^2}{\mu r_{min}} - 2\gamma\right]$$

$$r_{min} = \frac{r_0}{1+\varepsilon} = \frac{\ell^2/\mu\gamma}{1+\varepsilon}$$

$$E = \frac{\mu\gamma(1+\varepsilon)}{2\ell^2}\left[\gamma(1+\varepsilon) - 2\gamma\right]$$

$$= \frac{\mu\gamma^2}{2\ell^2}(\varepsilon+1)(\varepsilon-1) = \frac{\mu\gamma^2}{2\ell^2}(\varepsilon^2-1)$$

get ε in terms of E & ℓ
r_0 in terms of ℓ } DONE

Rocket Boosters

Changing orbit: $(E_1, \ell_1) \rightarrow (E_2, \ell_2)$ Imagine a tangential impulse at r_{min} (perigee)

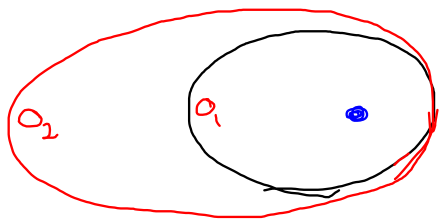
$$\frac{r_0^{(1)}}{1+\varepsilon_1} = \frac{r_0^{(2)}}{1+\varepsilon_2}$$ (perigees agree)

Say $\ell_2 = \lambda \ell_1$ ($v_2 = \lambda v_1$) ← thrust factor

$$r_0^{(1)} = \frac{\ell_1^2}{\mu\gamma} \qquad r_0^{(2)} = \frac{\ell_2^2}{\mu\gamma} = \lambda^2 r_0^{(1)}$$

$$\frac{r_0^{(1)}}{1+\varepsilon_1} = \frac{\lambda^2 r_0^{(1)}}{1+\varepsilon_2}$$

$\lambda > 1$, $\varepsilon_2 > \varepsilon_1$
$\lambda < 1$, $\varepsilon_2 < \varepsilon_1$ } modify eccentricity



$\lambda > 1$ in this case

Parting Words

In eliminating the cyclic/ignorable coordinate ℓ, we effectively went to a rotating frame

Non-inertial → next time we will consider general treatment of non-inertial ref. frames.