HW#6 Solutions

1) a) $U = kr^n \Rightarrow F = -knr^{n-1}\hat{r}$ balance forces $\frac{mv^2}{r} = knr^{n-1} \Rightarrow mv^2 = knr^n$

$$\boxed{T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}knr^n = \tfrac{n}{2}U}$$

b) $G = \vec{r}\cdot\vec{p}$ $\frac{dG}{dt} = \dot{\vec{r}}\cdot\vec{p} + \vec{r}\cdot\dot{\vec{p}} = 2T + \vec{F}\cdot\vec{r}$ $\frac{1}{t}\int_0^t \frac{dG}{dt} = 2\langle T\rangle + \langle \vec{F}\cdot\vec{r}\rangle$ by Mean Value Theorem

c) $\langle \frac{dG}{dt}\rangle = 0$ for large enough t Because: $\vec{r}\cdot\vec{p}$ is bounded for a periodic orbit, so $\frac{G(t)-G(0)}{t} \xrightarrow{t\to\infty} 0$

Thus: $2\langle T\rangle = -\langle \vec{F}\cdot\vec{r}\rangle = +\langle \vec{\nabla}U\cdot\vec{r}\rangle = \langle nkr^{n-1}\hat{r}\cdot(r\hat{r})\rangle = \langle nkr^n\rangle = n\langle U\rangle$

So: $\boxed{\langle T\rangle = \frac{n}{2}\langle U\rangle}$

2) a) $\vec{L} = m\vec{r}_0\times\vec{v}_0$ = const. $E = \frac{1}{2}m\vec{v}_0^{\,2} + V(|\vec{r}_0|)$

b) Note that E is dependent only on the magnitudes of $\vec{v}_0$ and $\vec{r}_0$

$\vec{L} = m\vec{r}_0\times\vec{v}_0 = m\hat{u}\,r_0 v_0\sin\theta \rightarrow$ dep. only on angle between $\vec{r}_0$ & $\vec{v}_0$

Reason: $\vec{L}$ is conserved, both in magnitude & direction $\Rightarrow$ $\vec{r}_0, \vec{v}_0$ span a plane $\perp$ to $\vec{L}$ & remain in that plane

polar coordinates in that plane are most convenient

1 angle to specify physics

c) The angle θ is $\pi/2$ at this point, since $\dot{r}=0$ at that point, $\overset{\text{def. of minimum}}{\Rightarrow}$ $\vec{v}\cdot\vec{r}$ must be vanishing

d) $E = \frac{1}{2}mv^2 + V(r) \overset{r=r_{min}}{=} \frac{1}{2}m\left(\frac{|\vec{L}|}{mr_{min}}\right)^2 + V(r_{min}) \Rightarrow E = \frac{1}{2}\frac{|\vec{L}|^2}{mr_{min}^2} + V(r_{min})$ Solve for r_{min} in terms of $E, |\vec{L}|$!

3) a) V must have units of energy (e.g. $kg\frac{m^2}{s^2}$) so $[a][x^{2n}] = [a]\,m^{2n} = kg\frac{m^2}{s^2}$ $\Rightarrow \boxed{[a] = \frac{kg}{s^2}\frac{1}{m^{2n-2}}}$

b) $dx = v\,dt$, need $v(x)$ $E_{tot} = \frac{1}{2}mv^2 + ax^{2n} \Rightarrow v = \sqrt{\frac{2}{m}}\left[E_{tot} - ax^{2n}\right]^{1/2} = \sqrt{\frac{2E}{m}}\left[1 - \frac{a}{E}x^{2n}\right]^{1/2} = 0$ @ extrema of motion: $x = (E/a)^{1/2n}$

$$dx = \sqrt{\frac{2E}{m}}\left[1-\frac{a}{E}x^{2n}\right]^{1/2}dt \Rightarrow \sqrt{\frac{m}{2E}}\frac{dx}{\left[1-\frac{a}{E}x^{2n}\right]^{1/2}} = dt \Rightarrow \sqrt{\frac{m}{2E}}\left(\frac{E}{a}\right)^{\frac{1}{2n}}\frac{d\tilde{x}}{\left[1-\tilde{x}^{2n}\right]^{1/2}} = dt$$

extrema @ $\tilde{x} = \pm 1$

since 2 extrema, ½ period from one to the other

Now integrate both sides: $\frac{1}{2}T = \sqrt{\frac{m}{2E}}\left(\frac{E}{a}\right)^{1/2n}\int_{-1}^{1}\frac{d\tilde{x}}{\left[1-\tilde{x}^{2n}\right]^{1/2}} = \sqrt{\frac{m}{2a}}\left(\frac{E}{a}\right)^{\frac{1}{2}\left(\frac{1}{n}-1\right)}\frac{2\sqrt{\pi}\,\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}$

$$\boxed{T = 2\sqrt{\frac{2m\pi}{a}}\left(\frac{a}{E}\right)^{\frac{1}{2}-\frac{1}{2n}}\frac{\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}}$$

c) From the above eq. for T, $T \propto E^{\frac{1}{2n}-\frac{1}{2}}$, and exponent vanishes only for $n=1$, $V\propto x^2$ (SHO potential)