HW #5 Solutions

1) a) $\mathcal{L}$ must have units of energy $\Rightarrow [\lambda] = kg\,\frac{s^2}{m^2}$ $\quad [a] = m^{-1}$ since arg. of cosh must be dimensionless

$[K] = $ joules $\left(\frac{kg\, m^2}{s^2}\right)$ since cosh must be dimensionless

b) Because $\mathcal{L}$ does not depend explicitly on t.

c) Calculate generalized momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \lambda v^3$; $E = p\dot{q} - \mathcal{L} = \lambda v^4 - \left[\frac{\lambda v^4}{4} - K\cosh(ax)\right]$

$$\boxed{E = \frac{3}{4}\lambda v^4 + K\cosh(ax)}$$

d) $E_0 = \frac{3}{4}\lambda v_0^4 + K\cosh(ax_0) = K\cosh(ax_0)$ (for $v_0 = 0$)

$E_f = \frac{3}{4}\lambda v_f^4 + K\cosh(ax_f) = \frac{3}{4}\lambda v_f^4 + K$ (for $x_f = 0$)

$$\boxed{v_f = \left[\frac{4K(\cosh(ax_0) - 1)}{3\lambda}\right]^{1/4}}$$

2) a) c has units $\frac{E}{m^2}$, or $\frac{kg}{s^2}$

b) 2, $x + y$!

c) $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{c(x+2y)^2}{10}$

d) Yes, since $\mathcal{L}$ is not explicitly dep. on t. $p_x = m\dot{x}$ $\quad p_y = p\dot{y}$ $\quad E = m\dot{x}^2 + m\dot{y}^2 - \mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{c(x+2y)^2}{10}$

e) $\mathcal{L}$ does not dep. on the combo $2x - y$ (lines $\perp$ to $x + 2y = $ const)

use new coord $q_1 = \frac{x+2y}{\sqrt{5}}$

$q_2 = \frac{2x-y}{\sqrt{5}}$

$$\boxed{\mathcal{L} = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{c q_1^2}{2}}$$

q_2 is cyclic/ignorable

f) $\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = p_{cyclic} = m\dot{q}_2$ (momentum along lines $\perp$ to $x + 2y = $ const.)

g) EoM $\quad \frac{\partial \mathcal{L}}{\partial q_2} = $ const. $\qquad \frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = -cq_1 - m\ddot{q}_1 = 0$

$m\dot{q}_2 = $ const $= p_2^0$

$$\boxed{q_2 = q_2^0 + \frac{p_2^0}{m}t} \qquad q_1 = A\cos(\omega t + \delta) \qquad \omega = \sqrt{\frac{c}{m}} \qquad \boxed{\begin{aligned} q_1^{(0)} &= A\cos\delta \\ \dot{q}_1^{(0)} &= -\omega A\sin\delta \end{aligned}}$$

h) Now $\mathcal{L}$ has no cylic coordinates (no $V(x,y) = $ const. lines in the plane)

$\Rightarrow$ No conservation laws (besides energy)

3) a) m is constrained to lie along line w/ x intercept z, slope $\tan\alpha$

$$\boxed{f(x,y,z) = y - (x - z)\tan\alpha}$$

b) $T = \frac{1}{2}m(\dot{x} + \dot{z})^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}M\dot{z}^2 \qquad U = mgy$

c) $\mathcal{L} = \frac{1}{2}m\left[(\dot{x} + \dot{z})^2 + \dot{y}^2\right] + \frac{1}{2}M\dot{z}^2 - mgy - \lambda\left[y - (x - z)\tan\alpha\right]$

d) EoM Constraint eq. imposes $y=(x-z)\tan\alpha$

$$\frac{\partial \mathcal{L}}{\partial x}-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}= \lambda \tan\alpha - m(\ddot{x}+\ddot{z})=0$$

$$\frac{\partial \mathcal{L}}{\partial y}-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}= -mg-\lambda - m\ddot{y}=0$$

$$\frac{\partial \mathcal{L}}{\partial z}-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}= -\lambda\tan\alpha - m(\ddot{x}+\ddot{z})-M\ddot{z}$$

$$2m(\ddot{x}+\ddot{z})+M\ddot{z}=0$$

$$-mg-\frac{m(\ddot{x}+\ddot{z})}{\tan\alpha}-m\ddot{y}=0$$

$$y=(x-z)\tan\alpha$$

e) We need 2 positions, 2 velocites $z_0, \dot{z}_0, x_0, \dot{x}_0$

4) a) just 1, angle rel. to vertical

b) $\mathcal{L}=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy$ dist. to bob from top

c) add $z=x_{top}$ as coord. $\dot{x}=\dot{z}+l\dot{\theta}\cos\theta$ $\dot{y}=l\dot{\theta}\sin\theta$ const: $z=a\cos\gamma t$

$$\mathcal{L}=\frac{1}{2}m\left[(\dot{z}+l\dot{\theta}\cos\theta)^2+l^2\dot{\theta}^2\sin^2\theta\right]-mgl\cos\theta-\lambda(z-a\cos\gamma t)$$

d) I was lucky enough to write time in only in the constraint. This time dep. is a total derivative, however, and so can be removed

e) $\mathcal{L}=\frac{1}{2}m\left[\dot{z}^2+l^2\dot{\theta}^2+\underbrace{2l\dot{z}\dot{\theta}\cos\theta}_{\text{mixed term}}\right]-mgl\cos\theta-\lambda(z-a\cos\gamma t)$

$\frac{d}{dt}(\dot{z}\sin\theta)=\ddot{z}\sin\theta+\dot{z}\dot{\theta}\sin\theta$ $\mathcal{L}'=\mathcal{L}-\frac{d}{dt}\left[ml\dot{z}\sin\theta\right]=\frac{1}{2}m\left[\dot{z}^2+l^2\dot{\theta}^2-2\ddot{z}\sin\theta\right]-mgl\cos\theta-\lambda(z-a\cos\gamma t)$

mixed term eliminated!

Since z can be eliminated via constraint eq., we have "norma" form for $\mathcal{L}$.