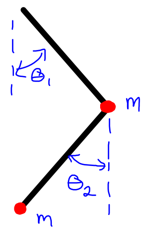
HW # 4 Solutions

1) Double Pendulum

a) There are 2 DoF, the two angles θ_1 and θ_2 that the pendula subtend relative to the vertical



b) $T_1 = \frac{1}{2} m v_1^2 = \frac{1}{2} m (l\dot{\theta}_1)^2$ $\qquad v_1^x = l\dot{\theta}_1 \cos\theta_1 \quad v_1^y = l\dot{\theta}_1 \sin\theta_1$

$T_2 = \frac{1}{2} m v_2^2$ $\qquad v_2^x = v_1^x + l\dot{\theta}_2 \cos\theta_2 \quad v_2^y = v_1^y + l\dot{\theta}_2 \sin\theta_2$

$$= \frac{1}{2} m l^2 \left[(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2)^2 + (\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2)^2\right]$$

$$= \frac{1}{2} m l^2 \left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

$U_1 = m g y_1 = mgl(1-\cos\theta_1)$

$U_2 = m g y_2 = mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2)$

$\Big\}$ eliminating constants, $U_{tot} = -mgl\left[2\cos\theta_1 + \cos\theta_2\right]$

$$\boxed{\mathcal{L} = \frac{1}{2} m l^2 \left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] + mgl\left[2\cos\theta_1 + \cos\theta_2\right]}$$

c) EoM

$$\frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = -ml^2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - 2mgl\sin\theta_1 - \frac{d}{dt}\underbrace{\left[2ml^2\dot{\theta}_1 + ml^2\dot{\theta}_2\cos(\theta_1-\theta_2)\right]}_{ml^2\left[2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) - \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right]} = 0$$

$$\boxed{-2mgl\sin\theta_1 - ml^2\left[2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right] = 0}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = ml^2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) - mgl\sin\theta_2 - \frac{d}{dt}\underbrace{\left[ml^2\dot{\theta}_2 + ml^2\dot{\theta}_1\cos(\theta_1-\theta_2)\right]}_{ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1\cos(\theta_1-\theta_2) - ml^2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)]}$$

$$= \boxed{-mgl\sin\theta_2 - ml^2\left[\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2)\right] = 0}$$

d) Small angle approx Only keep terms linear in angles

$$\boxed{\begin{aligned} -2mgl\theta_1 - ml^2\left[2\ddot{\theta}_1 + \ddot{\theta}_2\right] &= 0 \\ -mgl\theta_2 - ml^2\left[\ddot{\theta}_1 + \ddot{\theta}_2\right] &= 0 \end{aligned}}$$

Matrix Equation $\quad ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\ddot{\Theta} = -\omega_0^2\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\Theta \qquad \omega_0^2 \equiv \frac{g}{l}$$

$$\ddot{\Theta} = -\omega_0^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\Theta = -\omega_0^2\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\Theta = -\omega_0^2\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\Theta$$

Final Eq. $\quad \ddot{\Theta} = -\frac{g}{l}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\Theta$

Guess $\Theta = \begin{pmatrix} A \\ B \end{pmatrix}\cos(\omega t + \delta) \qquad -\omega^2\begin{pmatrix} A \\ B \end{pmatrix} = -\omega_0^2\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$

Need eigenvalues / eigenvectors

Eigenvalues: $2 \pm \sqrt{2} \Rightarrow \omega^2 = \omega_0^2(2 \pm \sqrt{2})$

Eigenvectors: $\begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}$

$$\boxed{\text{Final Solution: } \theta = \lambda_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos\left[\omega_0 \sqrt{2+\sqrt{2}} + \delta_1\right] + \lambda_2 \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \cos\left[\omega_0 \sqrt{2-\sqrt{2}} + \delta_2\right]}$$

2) a) The DoF of this system are the angle that the spring subtends, relative to the vertical, and the length r of the spring... that makes 2 DoF's.

b) $T = \frac{1}{2} m r^2 \dot\theta^2 + \frac{1}{2} m \dot r^2$

$= \frac{1}{2} m (\ell_0\lambda + \ell_0)^2 \dot\theta^2 + \frac{1}{2} m \ell_0^2 \dot\lambda^2$

$= \frac{1}{2} m \ell_0^2 \left[(1+\lambda)^2 \dot\theta^2 + \dot\lambda^2\right]$

$U = -mrg\cos\theta + \frac{1}{2} k (r - \ell_0)^2$

$= -gm\ell_0(\lambda+1)\cos\theta + \frac{1}{2} k \ell_0^2 \lambda^2$

$$\boxed{\mathcal{L} = \frac{1}{2} m \ell_0^2 \left[(1+\lambda)^2 \dot\theta^2 + \dot\lambda^2\right] - \frac{1}{2} k \ell_0^2 \lambda^2 + m \ell_0 g (\lambda+1)\cos\theta}$$

c) EoM

$$\frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\lambda}\right) = m\ell_0^2 (1+\lambda)\dot\theta^2 - k\ell_0^2 \lambda + mg\ell_0 \cos\theta - m\ell_0^2 \ddot\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = -m\ell_0 g \sin\theta - m\ell_0^2 \frac{d}{dt}\left[(1+\lambda)^2 \dot\theta\right]$$

$$= -m\ell_0 g\sin\theta - 2m\ell_0^2 (1+\lambda)\dot\lambda\dot\theta - m\ell_0^2 (1+\lambda)^2 \ddot\theta$$

$$\boxed{\begin{aligned} &(1+\lambda)\dot\theta^2 - \omega_s^2 \lambda + \frac{g\cos\theta}{\ell_0} - \ddot\lambda = 0 \\ &-\frac{g}{\ell_0}\sin\theta - 2(1+\lambda)\dot\lambda\dot\theta - (1+\lambda)^2\ddot\theta = 0 \end{aligned}}$$

d) Small angle approx: $-\omega_s^2 \lambda + g/\ell_0 - \ddot\lambda = 0$

$-\frac{g\theta}{\ell_0} - \ddot\theta = 0$

Define $\tilde\lambda = \lambda - \frac{g}{\omega^2 \ell_0} = \lambda - \frac{mg}{k\ell_0}$

$\ddot{\tilde\lambda} = -\omega_s^2 \tilde\lambda$

$\ddot\theta = -\frac{g}{\ell_0}\theta = -\omega_p^2 \theta$

We solved these eqs in class already!

$$\boxed{\text{Solution: } \theta(t) = B\cos(\omega_p t + \delta_p); \quad \tilde\lambda = A\cos(\omega_s t + \delta_s)}$$