Lecture 05.2 - Physics 523

$\boxed{\text{More on the spinning hoop}}$ Recall $\mathcal{L} = \frac{1}{2}mR^2(\dot\theta^2 + \omega^2\sin^2\theta) - mgR(1-\cos\theta) \Rightarrow \ddot\theta = \sin\theta(\omega^2\cos\theta - g/R)$

Stationary at $\theta_{sp} = \pm\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$, $\underset{\text{bottom / top}}{\theta_{sp} = 0, \pi}$

* Only solution for $\omega^2 > g/R$! $|\cos\theta| < 1$

Stability analysis: $\ddot\theta$ for small θ (taylor expand RHS) $\ddot\theta = \theta(\omega^2 - g/R) + \mathcal{O}(\theta^2)$

$\ddot\theta < 0$ for $\omega^2 < g/R$ STABLE

$\ddot\theta > 0$ for $\omega^2 > g/R$ UNSTABLE

Now consider $\frac{g}{\omega^2 R} \lesssim 1$

$\theta_0 \ll 1 \Rightarrow 1 - \frac{\theta_0^2}{2} = \frac{g}{\omega^2 R}$

$\theta > \theta_0$, $-\frac{\theta_0^2}{2}$ drives sign, $-$ for $\theta > \theta_0$

STABLE

Stable stationary point at $\theta = 0$ splits into 3 S.P.'s, one unstable, 2 stable

↳ these types of phase transitions play key roles in many physics subfields → Chaotic dynamics

"bifurcation"

* As before, in vicinity of stable S.P.'s, eq. of motion is approx $\ddot\varepsilon = -\omega^2\varepsilon$, w/ ω being a characteristic freq. of the stationary pt.

$\frac{\partial\mathcal{L}}{\partial q_i} \sim F_i$ (generalized force)

$\frac{\partial\mathcal{L}}{\partial \dot q_i} \sim p_i$ (generalized momentum)

$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \dot q_i}$

$\boxed{F_i = \dot p_i}$

$\mathcal{L}$ indep. of $q_i \Rightarrow$ p_i is conserved

↳ said that coord. q_i is "ignorable" or "cyclic"

Goal: express problem in terms of cyclic coordinates (if available)

$\boxed{\text{Symmetry and Conservation Laws}}$ If $\mathcal{L}$ does not depend on q_i, only $\dot q_i$, then $\mathcal{L}$ is unchanged by $q_i \to q_i' + \underset{\text{constant}}{\alpha}$

$\dot q_i \to (q_i' + \alpha)^{\cdot} = \dot q_i'$ → $\mathcal{L}$ is symmetric with respect to such transformations

there is a corresponding conserved quantity $\frac{\partial\mathcal{L}}{\partial \dot q_i}$!

$\boxed{\text{Example}}$ Recall again a particle in 3D, under the influence of a uniform gravitational field

$\mathcal{L} = T - U = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) - mgz$ x + y are cyclic/ignorable

Situation is clearly symmetric under shifts in x, y $\Rightarrow$ p_x + p_y are conserved

★ **Aside:** This correspondence is one of the most important notions in modern physics → i.e. conservation of electric charge

there is a symmetry principle underlying this!

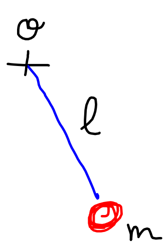
$\boxed{\text{Lagrange Multipliers}}$ As I mentioned before, there is a systematic way to approach constraints

→ in Lagrangian mechanics, can always ignore constraint forces (i.e. normal forces, string tensions, etc.)

Consider 2D system constrained to 1D (i.e. planar pendulum or bead on wire)

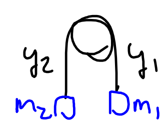
Construct a constraint equation $f(x,y) = \alpha$ (const)

Example planar pendulum



Constraint equation $f(x,y) \rightarrow x^2 + y^2 = \ell^2$ Mass constrained to circle of radius ℓ about origin

Remember the pulley example from Tuesday:



$y_1 + y_2 = L$ is the constraint equation

Consider a modified $\mathcal{L}$, with a new generalized coordinate $\lambda(t)$

in unconstrained coord.

$$\mathcal{L} = \underbrace{T - U}_{\mathcal{L}_0} + \underbrace{\lambda(t)\left(f(x,y) - \alpha\right)}_{\text{new term}}$$

Now have 3 E-L eq's

$$\frac{\partial \mathcal{L}_0}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \dot{x}} + \lambda \frac{\partial f}{\partial x} = 0 \quad (x \Leftrightarrow y)$$

$$\frac{\partial \mathcal{L}_0}{\partial \lambda} = f(x,y) - \alpha = 0$$ $\Rightarrow$ imposes constraint

Consider the pulley system

$$\mathcal{L} = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 - m_1 g y_1 - m_2 g y_2 + \lambda (y_1 + y_2 - L)$$

E-L eq's

$$\lambda - m_1 g - m_1 \ddot{y}_1 = 0$$
$$\lambda - m_2 g - m_2 \ddot{y}_2 = 0$$
$$y_1 + y_2 = L \Rightarrow \ddot{y}_1 = -\ddot{y}_2$$

* Note λ has units of Force

$$\lambda - m_1 g - m_1 \ddot{y}_1 = 0$$
$$\lambda - m_2 g + m_2 \ddot{y}_1 = 0$$
$$\ddot{y}_1 (m_1 + m_2) = g(m_2 - m_1)$$
$$\lambda (m_1 + m_2) = 2 g m_1 m_2 \qquad \lambda = 2g \frac{m_1 m_2}{m_1 + m_2}$$

↳ this is precisely F_t (string tension)

Using Lagrange multipliers, can extract constraint force!

$\boxed{\text{Now you do the pendulum}}$

$$f(x,y) = x^2 + y^2 = \ell^2$$

$$\mathcal{L} = \underbrace{\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)}_{T} + \underbrace{mgy}_{-U} + \lambda (x^2 + y^2 - \ell^2)$$

3 E-L eq's

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2\lambda x - m\ddot{x} = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = 2\lambda y + mg - m\ddot{y} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = x^2 + y^2 - \ell^2 = 0$$

$$2\lambda (x^2 + y^2) - m x \ddot{x} + mgy - my\ddot{y} = 0$$
$$\left(\frac{d^2}{dt^2} (x^2 + y^2) = 2x\ddot{x} + 2y\ddot{y} + 2\dot{x}^2 + 2\dot{y}^2 = 0 \right)$$
$$2\lambda \ell^2 + m(\dot{x}^2 + \dot{y}^2) + mgy = 0$$
$$2\lambda \ell + \frac{mv^2}{r} - mg\cos\theta = 0$$

Pendulum tension ✓

Conservation of Energy

$\mathcal{L} = T - U \rightarrow$ NOT conserved

$$\frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i; t) = \underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{= \frac{d}{dt} p_i} \dot{q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p} \ddot{q} + \frac{\partial \mathcal{L}}{\partial t}$$

Generally $\frac{d\mathcal{L}}{dt} = \sum_i \dot{p}_i \dot{q}_i + \sum p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} = \sum_i \frac{d}{dt}(p_i \dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}$

If $\mathcal{L}$ has no explicit time dependence, $\frac{\partial \mathcal{L}}{\partial t} = 0$, & $\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\sum_i (p_i \dot{q}_i)$

2T in cartesian

Define $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$

$2T - (T-U)$
$= T+U = E_{tot}$

$\frac{d}{dt}\mathcal{H} = 0$

$\mathcal{H}$ is the "Hamiltonian"

This quantity forms the backbone of Hamiltonian mechanics (and eventually quantum mech!