Lecture 05.1 - Physics 523

Hamilton's Principle Classical Trajectories minimize the Action $S = \int_{t_1}^{t_2} dt\, \mathcal{L}(\underbrace{q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N}_{\text{generalized coordinates}}; t)$

$\mathcal{L}$ is the Lagrangian function $\mathcal{L} = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q})$

Changing variables As promised, Lagrange's formulation makes transition between coordinate systems transparent

Say $\mathcal{L} = \mathcal{L}(\vec{r}(t), \dot{\vec{r}}(t); t)$ $\quad \vec{r}(t) = (x, y, z)$ Clearly, physics should not change if we, instead, $\vec{r}(t) = (r(t), \theta(t), \phi(t))$

BUT E-L equations are different! (but equivalent)

Cartesian	Polar
$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$	$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$
(y & z as well)	(θ + ϕ as well)

In cartesian: $\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} \Rightarrow$ the Force

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \Rightarrow$ the momentum

Note: don't have to take $\ddot{\vec{r}}$! (very messy in polar)

For general coordinates: $\frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow$ generalized force

$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow$ generalized momenta

} may not have units of force/momenta

Note: 6 "Force" components on rigid body
i) three components of force applied to body
ii) 3 torques corr. to 3 angles (axes)

$\boxed{\mathcal{F} = \dot{\pi}_q}$ $\quad \pi \to$ gen. momentum, $\mathcal{F} \to$ gen. force

Revisit 1 particle, 2D, polar coord: $\mathcal{L} = T - U = \frac{1}{2} m (\dot{\vec{r}})^2 - U(\vec{r})$ $\quad \vec{r} = r(t)\hat{r}(t)$

$$\dot{\vec{r}} = \dot{r}(t)\hat{r} + r(t)\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - U(r,\theta)$ 2 E-L equations, $\underbrace{r, \theta}_{\text{gen. coordinates}}$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} &= m r \dot{\theta}^2 - \frac{\partial U}{\partial r} - \frac{d}{dt}(m\dot{r}) = 0 \\ \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= -\frac{\partial U}{\partial \theta} - \frac{d}{dt}(m r^2 \dot{\theta}) = 0 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} m\ddot{r} - m r\dot{\theta}^2 &= -\frac{\partial U}{\partial r} \\ \frac{d}{dt}\underbrace{(m r^2 \dot{\theta})}_{L \text{ (angular momentum)}} &= \underbrace{-\frac{\partial U}{\partial \theta}}_{\text{torque!}} \end{aligned}}$$

can verify from $\vec{F} = -\vec{\nabla} U = -\left[\frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}\right]$

$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta}$ $\quad \tau = r F_\theta = -\frac{\partial U}{\partial \theta}$

Linear momentum eq's $\Leftrightarrow$ radial mom. + ang. mom.

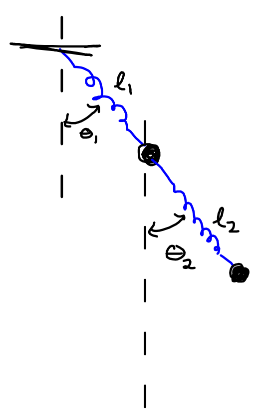
★ If $\mathcal{L}$ is indep. of q_i, E-L eq. is $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ $\quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is a conserved quantity!

Example Free particle in uniform gravitational field

x, y, z are gen. coord. $\quad \mathcal{L} = \frac{1}{2} m (\dot{\vec{r}})^2 - m g z$ $\quad$ ★ Indep. of x, y!

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}} &= m\dot{x} = p_x \\ \frac{\partial \mathcal{L}}{\partial \dot{y}} &= m\dot{y} = p_y \end{aligned}\right\} \text{both are conserved quantities!}$$

Example I Double spring pendulum



Calculating velocity is tough part

$T_1 = \frac{1}{2} m (\dot{\ell}_1^2 + \ell_1^2 \dot{\theta}_1^2)$

$\dot{\vec{r}}_1 = \dot{\ell}_1 \hat{r} + \ell_1 \dot{\theta}_1 \hat{\theta} = \dot{\ell}_1 (\sin\theta_1 \hat{x} - \cos\theta_1 \hat{y}) + \ell_1 \dot{\theta}_1 (\cos\theta_1 \hat{x} + \sin\theta_1 \hat{y})$

$= \hat{x}(\dot{\ell}_1 \sin\theta_1 + \ell_1\dot{\theta}_1 \cos\theta_1) + \hat{y}(\ell_1 \dot{\theta}_1 \sin\theta_1 - \dot{\ell}_1 \cos\theta_1)$

$$\left.\begin{aligned} &\dot{\vec{r}}_2(\text{rel}) = \quad \theta_1 \Leftrightarrow \theta_2 \\ &\dot{\vec{r}}_2 = \dot{\vec{r}}_2^{\,rel} + \dot{\vec{r}}_1 \end{aligned}\right\} T_2 = \frac{1}{2} m (\dot{\vec{r}}_2^{\,rel} + \dot{\vec{r}}_1)^2 = T_1 + T_2^{rel} + m \dot{\vec{r}}_2^{\,rel} \cdot \dot{\vec{r}}_1$$

$T_{tot} = 2T_1 + T_2^{rel} + m \dot{\vec{r}}_2^{\,rel} \cdot \dot{\vec{r}}_1$

$= m(\dot{\ell}_1^2 + \ell_1^2 \dot{\theta}_1^2) + \frac{m}{2}(\dot{\ell}_2^2 + \ell_2^2\dot{\theta}_2^2) + m(\dot{\ell}_1 \sin\theta_1 + \ell_1\dot{\theta}_1 \cos\theta_1)(1 \Leftrightarrow 2) + (\ell_1 \dot{\theta}_1 \sin\theta_1 - \dot{\ell}_1 \cos\theta_1)(1 \Leftrightarrow 2)$

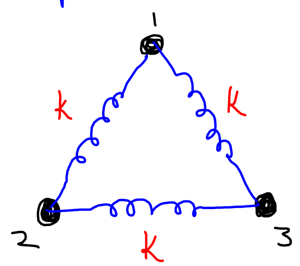
$= m(\dot{\ell}_1^2 + \ell_1^2 \dot{\theta}_1^2) + \frac{m}{2}(\dot{\ell}_2^2 + \ell_2^2\dot{\theta}_2^2) + m\left[(\dot{\ell}_1 \dot{\ell}_2 + \ell_1\dot{\theta}_1 \ell_2 \dot{\theta}_2)\cos(\theta_1 - \theta_2) + (\dot{\ell}_1 \ell_2 \dot{\theta}_2 - \ell_1 \dot{\theta}_1 \dot{\ell}_2)\sin(\theta_1 - \theta_2)\right]$

U is easier: $U_{tot} = -mgy_1 - mgy_2 + \frac{1}{2}k(\ell_1 - \ell_0)^2 + \frac{1}{2}k(\ell_2 - \ell_0)^2 = -mg\left[2\ell_1\cos\theta_1 + \ell_2\cos\theta_2\right] + \frac{1}{2}k\left[(\ell_1 - \ell_0)^2 + (\ell_2 - \ell_0)^2\right]$ Done!

4 E-L equations $\quad 2m\ell_1\dot{\theta}_1^2 + m\dot{\theta}_1 \ell_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \dot{\ell}_2 \sin(\theta_1 - \theta_2) + 2mg\cos\theta_1 + k(\ell_1 - \ell_0) - \frac{d}{dt}\left[2m\dot{\ell}_1 + m\dot{\ell}_2\cos(\theta_1 - \theta_2) + \ell_2\dot{\theta}_2 \sin(\theta_1 - \theta_2)\right] = 0$

(and 3 more ☹ equations)

Example II (neglect gravity)



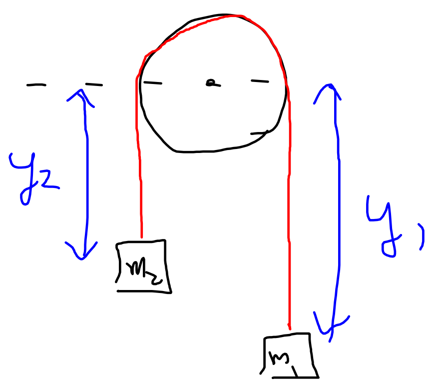
$$T=\frac{1}{2}m\left(\dot{\vec r}_1^{\,2}+\dot{\vec r}_2^{\,2}+\dot{\vec r}_3^{\,2}\right)$$

$$U=\frac{1}{2}k\left[\left(|\vec r_1-\vec r_2|-\ell_0\right)^2+\left(|\vec r_2-\vec r_3|-\ell_0\right)^2+\left(|\vec r_1-\vec r_3|-\ell_0\right)^2\right]$$

$$\mathcal{L}=T-U$$

Example III

$$y_1+y_2=L$$



$$T=\frac{1}{2}\left(m_1\dot y_1^2+m_2\dot y_2^2\right)=\frac{1}{2}\left(m_1\dot y_1^2+m_2\dot y_1^2\right)$$

$$U=-m_1gy_1-m_2gy_2=-m_1gy_1-m_2g(L-y_1)$$

$$=-g(m_1y_1-m_2y_1)-\underbrace{m_2gL}_{\text{const!}\ =0}$$

$$\mathcal{L}=\frac{1}{2}(m_1+m_2)\dot y_1^2+g(m_1y_1-m_2y_1)$$

EL eq's $\quad \frac{\partial\mathcal{L}}{\partial y_1}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot y_1}=\boxed{g(m_1-m_2)-(m_1+m_2)\ddot y_1=0}$

Constrained systems We have unconsciously taken constraints into account already. How can we be more systematic? Return to this later

e.g. rigid body has ∞'ly many subcomponents, w/ positions $\vec r$, but only 6 d.o.f.

$$\left.\begin{aligned}\vec r_\alpha&=\vec r_\alpha(q_1,\dots,q_n;t) &&\text{where } \alpha=1,\dots,N\\ q_i&=q_i(\vec r_1,\dots,\vec r_N;t) &&i=1,\dots,n\end{aligned}\right\}$$

★ in general, $DN\neq n$
↑ # of dimensions

Simple pendulum $\quad \vec r=(x,y)=(\ell\sin\theta,\ \ell\cos\theta)$
↑ constant $\rightarrow n=1$
$2N=2$

Some definitions

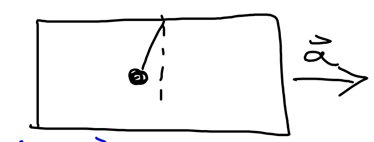
Holonomic systems: # of d.o.f. = # generalized coord — e.g. pendulum, mass on springs, etc.

a non-holonomic system: rubber ball on plane — d.o.f $= 2$ (x, y)

gen. coord. $(x, y, \underbrace{\theta, \phi, \phi'}_{\text{3 angles for rigid body}})$

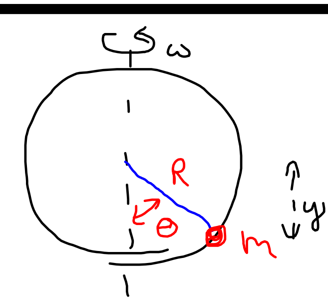
Non-holonomic systems require special consideration $\rightarrow$ not this class

Natural coord.: Relation between q's + $\vec{r}$'s indep. of t — e.g. pendulum in accel. car



$\vec{r} = (l\sin\theta + \frac{1}{2}at^2,\ l\cos\theta)$

Example: bead on spinning wire hoop



$T = \frac{1}{2}m(\dot{\vec{r}})^2 = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta\right)$

$\dot{\vec{r}} = R\dot{\theta}\hat{\theta} + \omega R\sin\theta\hat{\phi}$

$U = -mgy = -mgR\cos\theta$

$$\mathcal{L} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta$$

E-L eqs

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta - \frac{d}{dt}\left(mR^2\dot{\theta}\right) = 0$$

$$\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$$

Stationary points?

1) $\cos\theta = \frac{g}{\omega^2R}$ — Stable

2) $\sin\theta = 0$ — Unstable