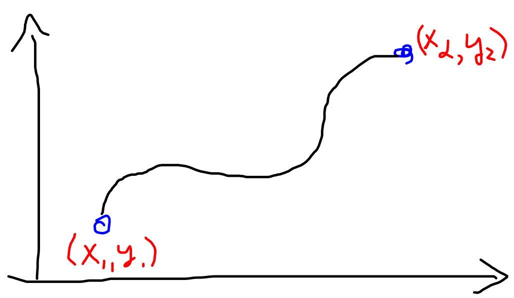
Lecture 04.2 - Physics 523

Variational Calculus Recall $\vec{F}=m\vec{a}$ is tricky in non-cartesian coordinates → would like more widely applicable formalism

Euler-Lagrange equations → provide EoM in generalized coordinates

What is variational calc.? What is the shortest path between 2 points? → straight line, of course Prove it!



$$d\ell=\sqrt{dx^2+dy^2}=\sqrt{dx^2+\left(\frac{dy}{dx}\right)^2dx^2}=dx\sqrt{1+(y')^2}$$

$$L=\int_{x_1}^{x_2}dx\,(1+y'^2)^{1/2}$$

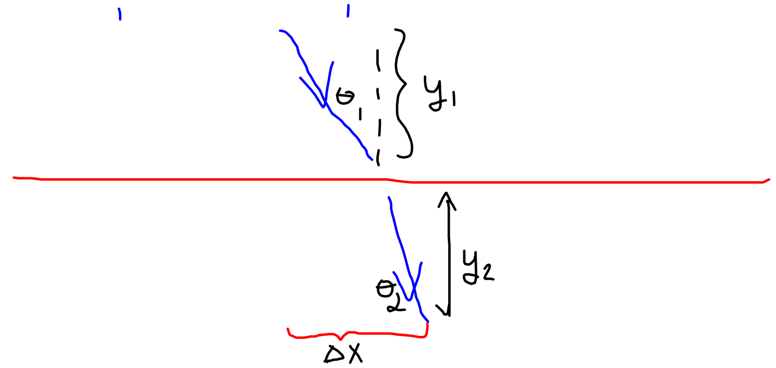
Want the function $y(x)$ that minimizes L

Your HW Fermat's principle

$$T=\int_1^2 dt=\int_1^2\frac{ds}{v}=\frac{1}{c}\int_1^2 n\,ds$$

index of refraction

Assume straight line in uniform medium



$$T=\frac{1}{c}\left[n_1\sqrt{x_1^2+y_1^2}+n_2\sqrt{x_2^2+y_2^2}\right]=\frac{1}{c}\left[n_1\sqrt{x_1^2+y_1^2}+n_2\sqrt{(\Delta x-x_1)^2+y_2^2}\right]$$

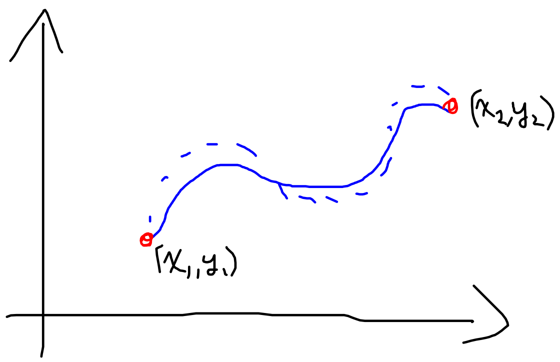
$$\frac{\partial T}{\partial x_1}=\frac{1}{c}\left[n_1\frac{x_1}{\sqrt{x_1^2+y_1^2}}-n_2\frac{(\Delta x-x_1)}{\sqrt{(\Delta x-x_1)^2+y_2^2}}\right]=\frac{1}{c}\left[n_1\frac{x_1}{\ell_1}-n_2\frac{x_2}{\ell_2}\right]$$

$$=\frac{1}{c}\left[n_1\sin\theta_1-n_2\sin\theta_2\right]=0$$

Snell's law: $\boxed{\dfrac{\sin\theta_1}{\sin\theta_2}=\dfrac{n_2}{n_1}}$

Euler-Lagrange Eq. Consider an integral over an undetermined integrand $S=\int_{x_1}^{x_2}f[y(x),y'(x),x]\,dx$

f is a function of the "trajectory" $y(x)$ and its derivative Eg. $ds=dx\underbrace{\sqrt{1+(y')^2}}_{f[y,y',x]}$



Need to systematically explore trajectories

Pretend $y_0(x)$ is solution, and now consider small variation in path

$$y(x) = y_0(x) + \underbrace{\alpha}_{\text{small }\#}\, \overbrace{\eta(x)}^{\text{arb. function w/ } \eta(x_1)=\eta(x_2)=0}$$

$$S_\eta(\alpha) = \int_{x_1}^{x_2} f[y, y', x]\, dx = \int_{x_1}^{x_2} f[y_0+\alpha\eta,\, y_0'+\alpha\eta',\, x]\, dx$$

If y_0 is solution, $\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} = 0$

$$\frac{\partial S_\eta(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right]dx = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta - \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta\right]dx + \underbrace{\left.\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2}}_{0 \text{ b.c. of } \eta(x_1)=\eta(x_2)=0}$$

So $\frac{\partial S_\eta}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\eta\, dx = 0$ if y_0 is a sol'n!! Must be true for all η. Integrand must be 0

$$\boxed{\text{Euler Lagrange eq's: } \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

Straight line? $f = \sqrt{1+(y')^2}$

$\frac{\partial f}{\partial y} = 0$ $\quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$ $\quad \frac{d}{dx}\frac{\partial f}{\partial y'} = y''\left[\frac{1}{\sqrt{\ }} - \frac{y'}{(\sqrt{\ })^3}\right] = 0$

$y'' = 0 \Rightarrow$ Straight line!

$\boxed{\text{The Brachistochrone}}$ "Shortest time" Add uniform gravity $\rightarrow$ frictionless tunnel from San-Fran to NYC How to build it?

$T = \int_1^2 \frac{ds}{v}$ $\quad v = \sqrt{2gy}$ ← dist below ground $\quad$ Want $y(x)$ $\quad ds = \sqrt{1+(y')^2}\, dx$, as usual

$$T = \int_{x_1}^{x_2}\sqrt{\frac{1+(y')^2}{2gy}}\, dx = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1+x'^2}{y}}\, dy$$

Euler-Lagrange: $\frac{\partial f}{\partial x} = 0$ $\quad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}}$ $\quad \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \Rightarrow \boxed{\frac{x'^2}{y(1+x'^2)} = \underset{\frac{1}{2a}}{\text{const}}}$ $\quad x' = \left[\frac{y}{2a-y}\right]^{1/2}$ $\quad$ Variables separated... Integrate!

So have $x = \int \sqrt{\frac{y}{2a-y}}\,dy$ Substitute $y = a(1-\cos\theta)$ $\quad x = a\int(1-\cos\theta)\,d\theta$

$= a(\theta - \sin\theta) + \text{const.}$

at $\theta = 0$, $y = 0$ (our initial condition)

$x = \text{const.} \Rightarrow \text{const} = 0$ for $x_0 = y_0 = 0$

Parametric eq. $\boxed{\begin{array}{l} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta)\end{array}}$ $\quad \theta = 2\pi \Rightarrow y$ is ϕ again, $x = 2\pi a$ $\Rightarrow L_{tot} = 2\pi a$ $\quad \boxed{a = \frac{L_{tot}}{2\pi}}$

time to go across: $\frac{T}{2} = \frac{1}{\sqrt{2g}}\int_0^{2a}\left[\frac{\frac{y}{2a-y}+1}{y}\right]^{1/2} dy = \sqrt{\frac{a}{g}}\,\pi$

$$\boxed{T = 2\pi\sqrt{\frac{a}{g}} = \left[\frac{L_{tot}}{2\pi g}\right]^{1/2}}$$

☆ Period independent of starting point!
→ how to build an "ideal" pendulum

Aside: Note that we transitioned between $y(x) \Leftrightarrow x(y)$ for convenience. This is quite a common techique. t is also often the indep. variable in mechanics problems

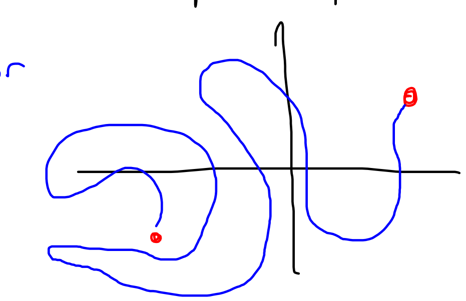
Also here we were interested in minimizing T. What about local minima? Maxima? In our later applications, extrema (stationary pts.) are all that matter.

Multiple dependent var.'s

Quantity of interest is often something like $S = \int dt\, f(q_1(t), \ldots, q_n(t); q_1'(t), \ldots, q_n'(t); t)$

q's represent the relevant degrees of freedom (e.g. 3 cm coordinates, 3 angles giving orientation of single rigid body)
6 total q's!

As an intro. to this: I lied to you. We assumed, in our proof that straight lines are the shortest path between pairs of pts. that path has form $x = x(y)$ Doesn't work for (multi-valued!)



Our path in the plane really has 2 variables, $x(u)$ and $y(u)$ (parametrizes point on 1D path) $\quad L = \int_{u_1}^{u_2}\sqrt{x'(u)^2 + y'(u)^2}\,du$

Consider right path, $x_0(u)$ and $y_0(u)$ and slightly deform this correct trajectory: $\quad x(u) = x_0(u) + \alpha\,\xi(u)$

$y(u) = y_0(u) + \beta\,\eta(u)$

Minimization requirement: $\frac{\partial L}{\partial\alpha} = \frac{\partial L}{\partial\beta} = 0$

This yields 2 different Euler–Lagrange Equations For $S = \int f(x,y,x',y';u)\,du$

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0 \qquad \frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0}$$

Straight Line (No lies this time)

$$\left.\begin{aligned} \frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} &= -\frac{d}{du}\frac{x'}{\sqrt{(x')^2+(y')^2}} = 0 \\ \frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} &= -\frac{d}{du}\frac{y'}{\sqrt{x'^2+y'^2}} = 0 \end{aligned}\right\} \quad \frac{x'}{\sqrt{x'^2+y'^2}} = C_1 \qquad \frac{y'}{\sqrt{x'^2+y'^2}} = C_2 \qquad \frac{dy}{dx} = \frac{C_2}{C_1} \equiv m$$

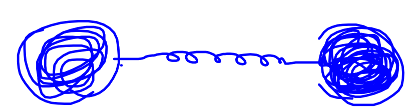
$$y = mx + b$$

General case w/ many variables For N variables, N equations

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}_i} = 0}$$

The Lagrangian For many physical systems, there is a special f where E-L equations are precisely the mechanical Equations of motion!

Generalized coordinates Consider rigid "dumbell" connected by springs



Useful coordinates: $\vec{R}_{cm}$, $\underbrace{\vec{r}_1 - \vec{r}_2}$ orientation & separation dist. of masses

★ There is a function $\mathcal{L}(\vec{R}_{cm}(t), \Delta\vec{r}(t); t)$ such that extremization of $S = \int dt\, \mathcal{L}$ ← the Lagrangian yields the eq's of motion!

↑ the action

For general $\mathcal{L}(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_n;t)$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

★ The Lagrangian is (for a system w/ only conservative forces) $\boxed{\mathcal{L} = T - U}$ Consider single particle in 3D

$$\mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z};t) = \frac{1}{2}m(\dot{\vec{r}})^2 - U(\vec{r})$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \overbrace{-\frac{\partial U}{\partial x}}^{F_x} - \frac{d}{dt}\overbrace{(m\dot{x})}^{p_x} = 0 \rightarrow \boxed{F_x = \dot{p}_x}$$

Same for y, z E-L eq's $\boxed{\vec{F} = -\vec{\nabla} U = \dot{\vec{p}}}$

Get Newton's 2nd!