# Lecture 04.1 - Physics 523

**Oscillator Review** The S.H.O (simple harmonic oscillator) is ubiquitous

Near S.P., $U(x) = U(x_{sp}) + \frac{1}{2}(\Delta x)^2 U''(x_0)$

linear force law $F = -k\Delta x$ ; $k \leftarrow U''(x_0)$

$\ddot{x} = -\frac{k}{m}x$ ; $\frac{k}{m} \leftarrow \omega^2$

General Solution: $x(t) = C_+ e^{+i\omega t} + C_- e^{-i\omega t}$

Real solutions: $x(t) = X_0 \cos(\omega t + \delta)$

**Energy of the oscillator:**

$T = \frac{1}{2}m(\dot{x})^2 = \frac{1}{2}m\left[-X_0\omega \sin(\omega t+\delta)\right]^2$

$= \frac{1}{2}m x_0^2 \omega^2 \sin^2(\omega t+\delta)$

$U = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2)\left[x_0\cos(\omega t+\delta)\right]^2$

$= \frac{1}{2}m x_0^2\omega^2 \cos(\omega t+\delta)$

$E_{tot} = \frac{1}{2}m x_0^2\omega^2 \underbrace{\left[\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right]}_{=1}$

$= \frac{1}{2}m x_0^2 \omega^2 \rightarrow$ Constant (energy is indeed conserved)

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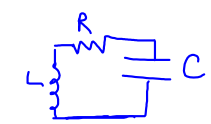
**Damped Oscillators** Consider a viscous drag force in add. to Hook's law $F = -kx - b\dot{x}$ (is there a U?)

Equation of motion: $\underbrace{m\ddot{x} + b\dot{x} + kx = 0}$ ☺ Linear Homogeneous

( Same formula arises in circuits $\underbrace{L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0}_{\text{Kirchoff's law}}$ )




$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ ; $2\beta \sim \frac{b}{m}$, $\omega_0^2 \leftarrow \frac{k}{m}$

if you can find 2 independent solutions, you're done (superposition does the rest)

guess: $x = e^{rt}$ $(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$ or $r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$ ← 2 solutions!

$x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$

2 cases:

1) $\beta > \omega_0 \rightarrow$ exponentials real. Only solutions are exponentially damped $x(t) = C_1 e^{-\beta_1 t} + C_2 e^{-\beta_2 t}$

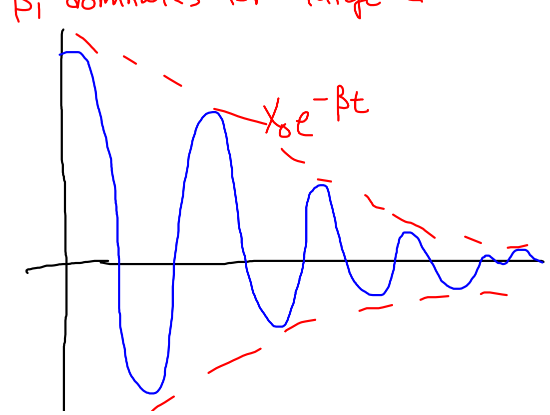
$\beta_1 = \beta\left[1 - \sqrt{\beta^2 - \omega_0^2}\right]$
$\beta_2 = \beta\left[1 + \sqrt{\beta^2 - \omega_0^2}\right]$

Referred to as "overdamped" oscillator Eg. car shocks

$\beta_1$ dominates for large t

2) $\beta < \omega_0 \rightarrow$ exponentials complex $x = e^{-\beta t}\left[C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right]$ $\omega_d = \sqrt{\omega_0^2 - \beta^2}$

$= e^{-\beta t} A\cos(\omega_d t - \delta)$ Exponentially damped oscillations




3) Critical Damping $\beta = \omega_0 \rightarrow$ solutions become degenerate

New sol'n $x(t) = e^{-\beta t}\left[C_1 + C_2 t\right]$

$\boxed{\text{Driving a damped oscillator}}$ Add a non-homogenous term! $m\ddot{x} + b\dot{x} + kx = F(t)$ ← external driving

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$ Can write this as $Dx = f$ ← only for linear!

$D = \frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \omega_0^2$

Note homogenous Sol'n is always part of total sol'n $x = x_h + x_{nh}$

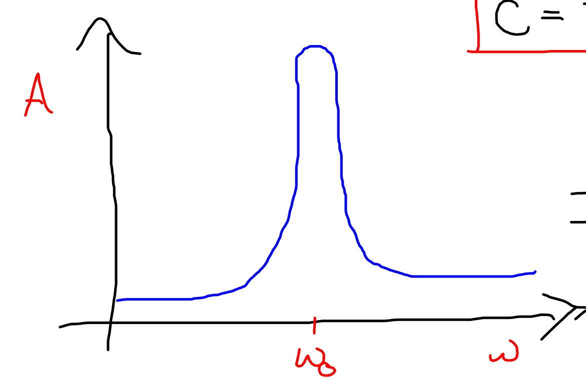
$$\left.\begin{aligned} Dx_h &= 0 \\ Dx_{nh} &= f \end{aligned}\right\} D(x_h + x_{nh}) = f$$

Consider sinusoidal force → $f(t) = f_0 e^{i\omega t}$ Consider complex $x(t) \equiv z(t)$ $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$ → $z = Ce^{i\omega t}$

$C(-\omega^2 + 2\beta\omega i + \omega_0^2) = f$

$$\boxed{C = \frac{f}{\omega_0^2 - \omega^2 + 2\beta\omega i}}$$

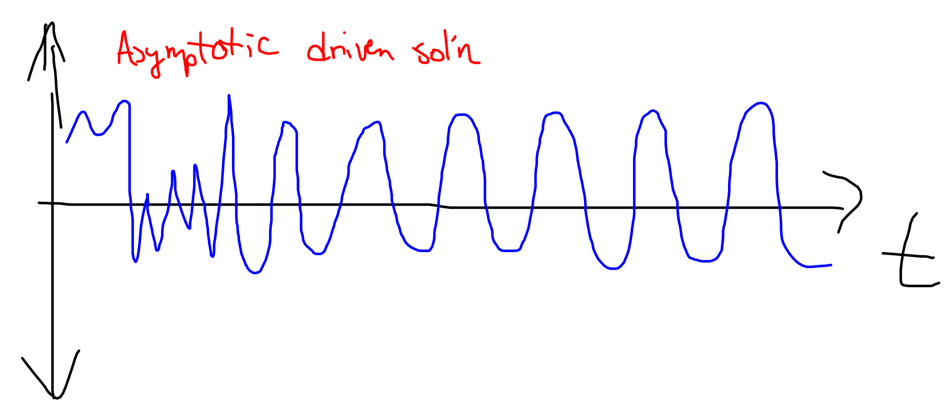
$C = Ae^{-i\delta}$ $\boxed{A = \frac{f_0}{[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2]^{1/2}}}$ $\boxed{\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)}$




If you drive any oscillator at right freq., you get resonant enhancement

Inital behavior displays messy looking behavior

Interference of $\underbrace{\text{exponentially damped sol'ns}}_{\text{homogenous sol'n}}$ w/ driven sol'n



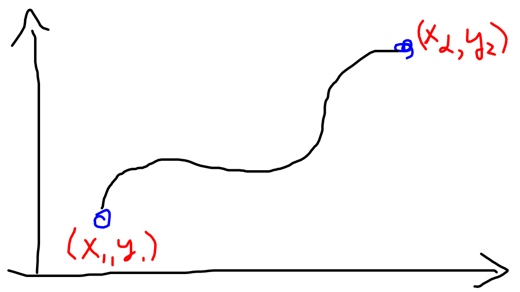
## Variational Calculus

Recall $\vec{F} = m\vec{a}$ is tricky in non-cartesian coordinates → would like more widely applicable formalism

Euler-Lagrange equations → provide EoM in generalized coordinates

### What is variational calc.?

What is the shortest path between 2 points? → straight line, of course. Prove it!




$$dl = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2} = dx\sqrt{1+(y')^2}$$

$$L = \int_{x_1}^{x_2} dx\,(1+y'^2)^{1/2}$$

Want the function $y(x)$ that minimizes $L$

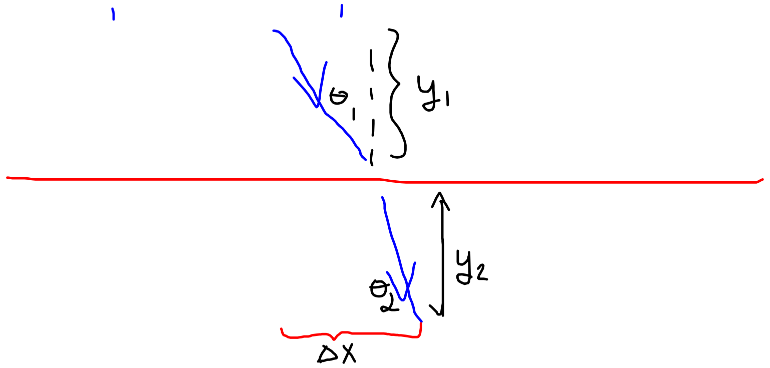
### Your HW

Fermat's principle

$$T = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\,ds$$

← index of refraction

Assume straight line in uniform medium




$$T = \frac{1}{c}\left[n_1\sqrt{x_1^2+y_1^2} + n_2\sqrt{x_2^2+y_2^2}\right] = \frac{1}{c}\left[n_1\sqrt{x_1^2+y_1^2} + n_2\sqrt{(\Delta x - x_1)^2 + y_2^2}\right]$$

$$\frac{\partial T}{\partial x_1} = \frac{1}{c}\left[n_1\frac{x_1}{\sqrt{x_1^2+y_1^2}} - n_2\frac{(\Delta x - x_1)}{\sqrt{(\Delta x - x_1)^2 + y_2^2}}\right] = \frac{1}{c}\left[n_1\frac{x_1}{l_1} - n_2\frac{x_2}{l_2}\right]$$

$$= \frac{1}{c}\left[n_1\sin\theta_1 - n_2\sin\theta_2\right] = \emptyset$$

Snell's law: $\boxed{\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{n_2}{n_1}}$

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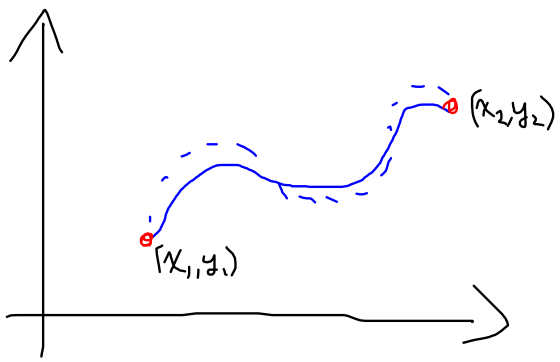
## Euler-Lagrange Eq.

Consider an integral over an undetermined integrand $S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx$

$f$ is a function of the "trajectory" $y(x)$ and its derivative. Eg. $ds = dx\underbrace{\sqrt{1+(y')^2}}_{f[y, y', x]}$




Need to systematically explore trajectories

Pretend $y_0(x)$ is solution, and now consider small variation in path

$$y(x) = y_0(x) + \underset{\uparrow \text{small } \#}{\alpha}\, \overset{\curvearrowright \text{arb. function w/ } \eta(x_1)=\eta(x_2)=0}{\eta(x)}$$

$$S_\eta(\alpha) = \int_{x_1}^{x_2} f[y, y', x]\, dx = \int_{x_1}^{x_2} f[y_0 + \alpha\eta, y_0' + \alpha\eta', x]\, dx$$

If $y_0$ is solution, $\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} = 0$

$$\frac{\partial S_\eta(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right] dx = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta - \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta\right] dx + \underbrace{\left.\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2}}_{0 \text{ b.c. of } \eta(x_1)=\eta(x_2)=0}$$

So $\frac{\partial S_\eta}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] \eta\, dx = 0$ if $y_0$ is a sol'n!! Must be true for all $\eta$ Integrand must be 0

$$\boxed{\text{Euler Lagrange eq's: } \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

Straight line? $f = \sqrt{1 + (y')^2}$

$\frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \quad \frac{d}{dx}\frac{\partial f}{\partial y'} = y''\left[\frac{1}{\sqrt{\phantom{x}}} - \frac{y'}{(\sqrt{\phantom{x}})^3}\right] = 0$

$y'' = 0 \Rightarrow$ Straight line!

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The Brachistochrone "Shortest time" Add uniform gravity $\rightarrow$ frictionless tunnel from San-Fran to NYC How to build it?

$T = \int_1^2 \frac{ds}{v} \quad v = \sqrt{2gy}$ ← dist below ground Want $y(x)$ $ds = \sqrt{1+(y')^2}\, dx$, as usual $T = \int_{x_1}^{x_2} \sqrt{\frac{1+(y')^2}{2gy}}\, dx = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2} \sqrt{\frac{1+x'^2}{y}}\, dy$

Euler-Lagrange: $\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}} \quad \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 \Rightarrow \boxed{\frac{x'^2}{y(1+x'^2)} = \text{const} = \frac{1}{2a}}$

$x' = \left[\frac{y}{2a-y}\right]^{1/2}$ Variables separated... Integrate!