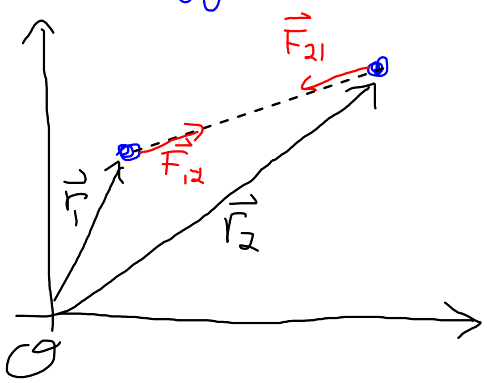
Lecture 03.2 - Physics 523

Energy of interaction In absence of external forces, in closed system w/ 2 particles have $\vec{F}_{12} = -\vec{F}_{21}$



$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1, \vec{r}_2)$ this force must be independent of choice of origin

$\rightarrow$ depends only on $\vec{r}_1 - \vec{r}_2$ $\quad \vec{F}_{12} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$

Example: gravity $\vec{F}_{12} = -\dfrac{G m_1 m_2 \hat{r}_{21}}{|\vec{r}_1 - \vec{r}_2|^2} = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$

unit vector from 2 to 1

Lets say we fix $\vec{r}_2 = 0$ at origin, then $\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1)$ if $\vec{F}_{12}$ is conservative, need $\vec{\nabla}_1 \times \vec{F}_{12}(\vec{r}_1) = \emptyset$

curl wrt. $\vec{r}_1$ coordinates

in cartesian: $\vec{\nabla}_1 = \hat{x}\dfrac{\partial}{\partial x_1} + \hat{y}\dfrac{\partial}{\partial y_1} + \hat{z}\dfrac{\partial}{\partial z_1}$

In this case, can define $U_1 = U_1(\vec{r}_1)$, w/ $\boxed{\vec{F}_{12}(\vec{r}_1) = -\vec{\nabla}_1 U(\vec{r}_1)}$ now restore the $\vec{r}_2$ dependence by shifting the origin

minus signs compensate

$\vec{F}_{12}(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$ also, $\vec{F}_{12} = -\vec{F}_{21} \rightarrow \vec{F}_{21} = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)$ Chain rule, $\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)$

Conservation of Energy $dT = dT_1 + dT_2 = W_{on\,1} + W_{on\,2} = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21} = (d\vec{r}_1 - d\vec{r}_2)\left[-\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)\right]$

$= d\vec{r} \cdot \left[-\vec{\nabla} U(\vec{r})\right]$ w/ $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

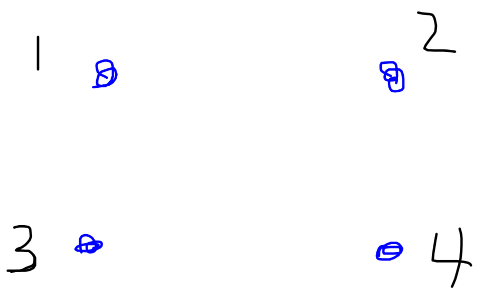
$= -dU$ (dr's cancel)

So: $dT = -dU$ and $dE = d(T+U) = dT + dU = 0$ (Energy is conserved, as expected)

$\boxed{E = T_1 + T_2 + U = \text{constant}}$

Energy of Multiparticle Systems

For no external forces, have $T_{tot} = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$



$U_{tot} = \underbrace{U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}}_{\text{don't count } U\text{'s twice!}}$

$U_{tot} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta}$

ensures against double-counting

$$\boxed{E_{tot} = \sum_\alpha T_\alpha + \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta} = \text{const}}$$

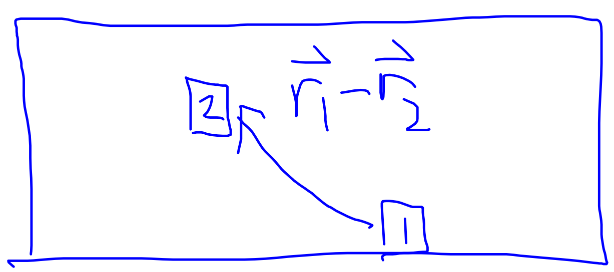
Adding external conservative force

pot. from ext. forces

internal potentials between particles

$E_{tot} = \sum_\alpha T_\alpha + \sum_\alpha U_\alpha^{ext} + \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}^{int}$ is conserved

Rigid bodies



$\vec{r}_1 - \vec{r}_2$ is fixed $\Rightarrow U_{12}$ never changes

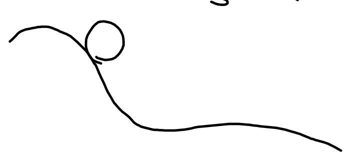
$$E_{tot} = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \sum_\alpha U_\alpha^{ext} + \underbrace{\sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}^{int}}_{\text{fixed, might as well be } \emptyset}$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \sum_\alpha U_\alpha^{ext}$$

$$= \frac{1}{2} M v^2 + U_{tot}^{ext}$$

Rolling stone

$I = \frac{2}{5} M R^2$



$T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} M v^2 + \frac{1}{2}\left(\frac{2}{5} M\right)\underbrace{(R\omega)^2}_{v^2} = \frac{7}{10} M v^2$

$U = Mgh$

height of center of sphere

Say $v_0 = 0$

$E = \frac{7}{10} M v^2 + Mgh$ constant

$$\boxed{Mg(h_0 - h_f) = -\frac{7}{10} M (v_0^2 - v_f^2)}$$

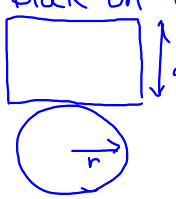
Hooke's Law

Many physical systems exhibit linear restorative force: $\underbrace{F = -k(x - x_{eq})}_{\text{force due to stretched / compressed spring}}$ ← $x = x_{eq}$ is stationary point

Note: 1D force dep. only on $x \Rightarrow$ conservative

Associated U: $U(x) = \frac{1}{2} k (x - x_{eq})^2$

Recall Block on cylinder: $U''(\theta = 0) = mg(r-b) =$ "k"



Restorative $F = -mg(r-b)\theta$

$(r > b)$

Crucial Aside

Can Taylor expand any potential in vicinity of stationary point: $U(x_0 + \Delta x) = U(x_0) + \Delta x \underbrace{U'(x_0)}_{} + \frac{1}{2}(\Delta x)^2 U''(x_0)$

can choose to be 0 $= -F(x_0) = 0$ at SP

$$= \frac{1}{2} \underbrace{U''(x_0)}_{k} (x - x_0)^2$$

$$= \frac{1}{2} k x^2$$

Nearly all systems are approximately springs!! (unless you go far from ⇄)

Simple Harmonic Motion

If you learn only one thing, learn this

$F = m\ddot{x} = -kx \qquad \ddot{x} = -\omega^2 x$ with $\omega^2 = \frac{k}{m}$

Solutions $\boxed{x = c_+ e^{+i\omega t} + c_- e^{-i\omega t}}$

(Also $A \sin \omega t + B \cos \omega t$)

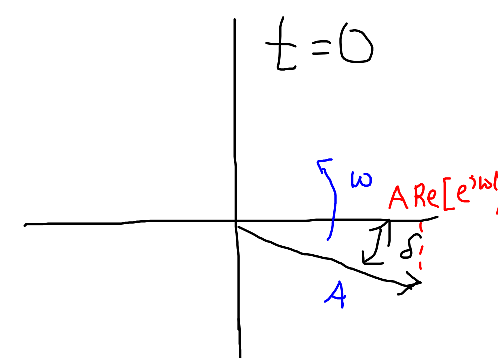
Arbitrary coefficients (initial conditions fix them)

Recall for linear eq.'s, have superposition principle

→ Solutions are periodic, w/ angular frequency ω

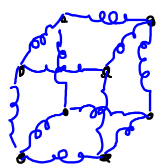
If we restrict to real-valued solutions, $c_- = (c_+)^*$ → $x = c_+ e^{i\omega t} + \text{c.c.} = 2\,\mathrm{Re}[\underbrace{c_+}_{\frac{A}{2} e^{-i\delta}} e^{i\omega t}] = A\,\mathrm{Re}[e^{i(\omega t - \delta)}]$

$$= A \cos(\omega t - \delta)$$



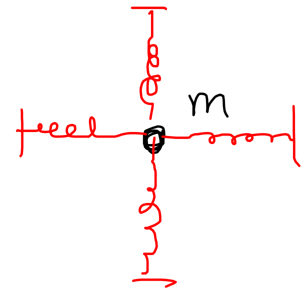
A 2D spring system

A very interesting system to study is a bunch of masses spread out on a lattice



repeated on ∞ lattice (eg, a crystal, w masses = atoms, atomic bonds hold it together

As a step towards this, consider 1 mass in 2D:



ends are fixed, same constant, $k/2$ for each spring

$F_x = -kx \quad F_y = -ky \quad \boxed{\vec{F} = -k\vec{r}}$

Equations of motion: $\ddot{x} = -\frac{k}{m}x = -\omega^2 x$ ($\omega = \sqrt{k/m}$), $\ddot{y} = -\omega^2 y$ $\Big\}$ $\ddot{\vec{r}} = -\omega^2 \vec{r}$

Note: k could be different for x & y directions (ω_x & ω_y)

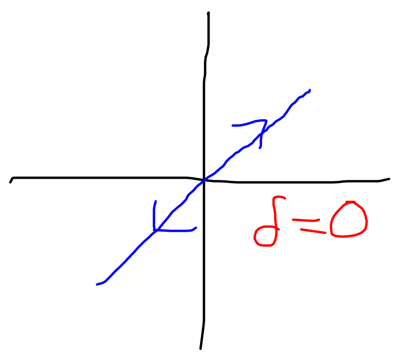
These are linear & uncoupled $\Rightarrow$ solve separately & superpose!

$$\boxed{\begin{aligned} x(t) &= X_0 \cos(\omega t - \delta_x) \\ y(t) &= Y_0 \cos(\omega t - \delta_y) \end{aligned}}$$

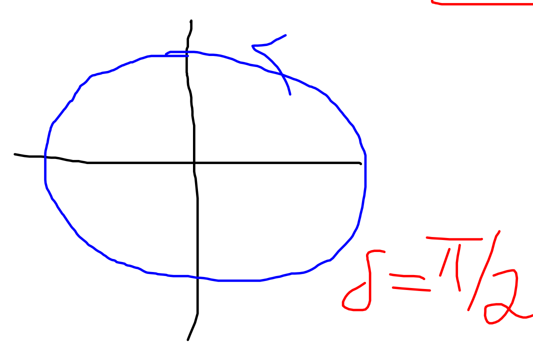
Pick $t=0$ s.t. $\delta_x = 0$

$\delta \equiv \delta_y$ is relative phase

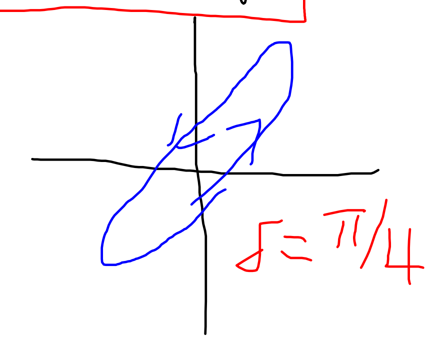
Types of motion:



line



circle



ellipse

What if $k_x \neq k_y$?

$x(t) = X_0 \cos(\omega_x t)$

$y(t) = Y_0 \cos(\omega_y t - \delta)$

behavior depends crucially on ω_x / ω_y

$\frac{\omega_x}{\omega_y}$ rational $\Rightarrow \frac{\omega_x}{\omega_y} = \frac{m}{n}$ integers, reduced fraction

$n\omega_x = m\omega_y$

when $t = \min(m,n) \cdot 2\pi$, motion repeats

But if $\frac{\omega_x}{\omega_y}$ is irrational (π, $\sqrt{2}$, etc.) then motion never repeats