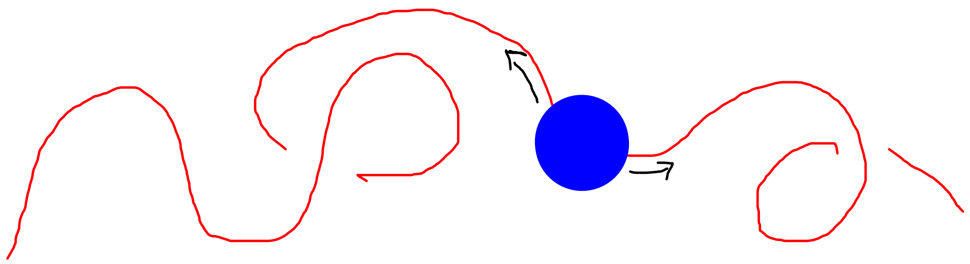
Lecture 03.1 - Physics 523

1D Motion

What do we mean by 1D? Simplest case: particle constrained to move in straight line

However: many systems can be completely specified w/ 1 #

e.g. bead on wire (dentist / doctors office specialty)



In absence of friction, E is conserved $E = T + U$ ← gravitational P.E.

$= \frac{1}{2}mv^2 + mgh$

← must be tangential to curve

Can parameterize system by τ (distance along wire) $h = h(\tau)$

trajectory $\tau = \tau(t)$

Note $\vec{F} = \vec{F}(\tau)$ (satisfies 1st of 2 req. for force to be "conservative")

→ Despite complicated path in 2D, problem is still 1D (always carefully count d.o.f.!)

other 1D problems: pendulum, complicated mess of pulleys + levers

---

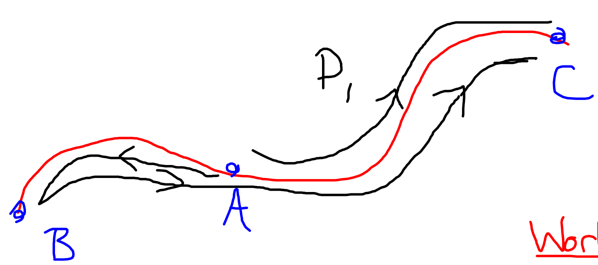
Aside: all forces w/ $\vec{F} = \vec{F}(\tau)$ are conservative

← single parameter

Recall: 1) $\vec{F} = \vec{F}(\vec{r})$

2) W is path independent

1D paths



P_1 goes directly from A to C

P_2 from A to B to C

Work: $W_1 = -\int_{\tau_A}^{\tau_C} F(\tau)\, d\tau$

Note $F(\tau)$ is comp. of force ∥ to path

$$W_2 = -\underbrace{\int_{\tau_A}^{\tau_B} F(\tau)\, d\tau - \int_{\tau_B}^{\tau_A} F(\tau)\, d\tau}_{\text{CANCEL}} - \int_{\tau_A}^{\tau_C} F(\tau)\, d\tau = W_1$$

Full EoM

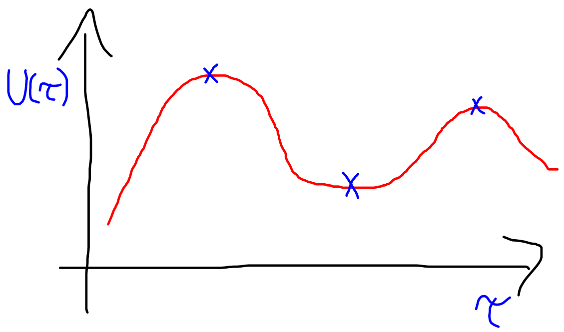
So for 1D forces $\vec{F}=\vec{F}(x)$, $E=T+U$ ← associated potential energy. Conserved $T=\frac{1}{2}mv^2=E_0-U(x)$ or $\boxed{\dot{x}=\pm\sqrt{\frac{2}{m}}\sqrt{E_0-U(x)}}$

Separate variables $dt=\frac{\pm dx}{\left[\frac{2}{m}(E-U(x))\right]^{1/2}}$ and integrate

---

Stationary points & stability

$\vec{F}=-\vec{\nabla}U \rightarrow F=-\frac{\partial U}{\partial x}$ for 1D problems $F=0$ when $\frac{\partial U}{\partial x}=0$



If $\dot{r}=0$, and r itself is chosen for $U'(r)=0$, then $\ddot{r}$ also vanishes, + no motion occurs

Stationary point

Stability, if, in nbh of stationary point, F is restorative (towards S.P.) then SP is stable

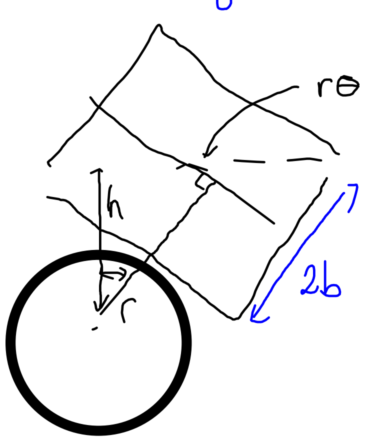
$U''(r_{sp})>0$ Stable S.P.

$U''(r_{sp})<0$ Unstable S.P.

Sounds trivial → study non-trivial example

Block on fixed cylinder

→ whole system specified by θ



height of CM

$h=(r+b)\cos\theta+r\theta\sin\theta$

$U(\theta)=mgh$

$U'(\theta)=mg[-(r+b)\sin\theta+r\sin\theta+r\theta\cos\theta]=mg[-b\sin\theta+r\theta\cos\theta]\overset{\theta=0}{=}0$

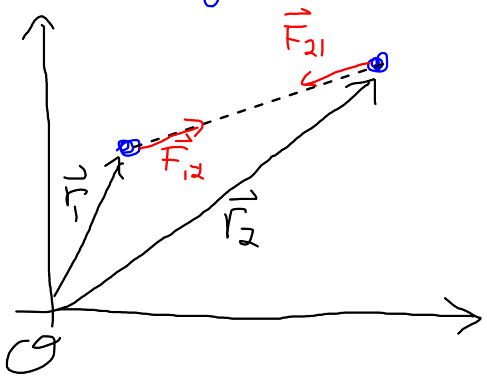
$U''(\theta)=mg[-b\cos\theta+r\cos\theta-r\theta\sin\theta]\overset{\theta=0}{=}mg(r-b)$

SP is stable for $r>b$, unstable $r<b$

how does h depend on θ

Energy of interaction

In absence of external forces, in closed system w/ 2 particles have $\vec{F}_{12} = -\vec{F}_{21}$



$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1, \vec{r}_2)$ this force must be independent of choice of origin

$\rightarrow$ depends only on $\vec{r}_1 - \vec{r}_2$ $\quad \vec{F}_{12} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$

Example: gravity $\vec{F}_{12} = -\dfrac{G m_1 m_2 \, \hat{r}_{21}}{|\vec{r}_1 - \vec{r}_2|^2} = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$

$\hat{r}_{21}$: unit vector from 2 to 1

Lets say we fix $\vec{r}_2 = 0$ at origin, then $\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1)$ if $\vec{F}_{12}$ is conservative, need $\vec{\nabla}_1 \times \vec{F}_{12}(\vec{r}_1) = \emptyset$

$\vec{\nabla}_1$: curl wrt. $\vec{r}_1$ coordinates

in cartesian: $\vec{\nabla}_1 = \hat{x}\dfrac{\partial}{\partial x_1} + \hat{y}\dfrac{\partial}{\partial y_1} + \hat{z}\dfrac{\partial}{\partial z_1}$

In this case, can define $U_1 = U_1(\vec{r}_1)$, w/ $\boxed{\vec{F}_{12}(\vec{r}_1) = -\vec{\nabla}_1 U(\vec{r}_1)}$ now restore the $\vec{r}_2$ dependence by shifting the origin

$\vec{F}_{12}(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$ also, $\vec{F}_{12} = -\vec{F}_{21} \rightarrow \vec{F}_{21} = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)$

Chain rule, $\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)$ (minus signs compensate)

Conservation of Energy

$$dT = dT_1 + dT_2 = W_{on\,1} + W_{on\,2} = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21} = (d\vec{r}_1 - d\vec{r}_2)\left[-\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)\right]$$

$$= d\vec{r} \cdot \left[-\vec{\nabla} U(\vec{r})\right] \quad \text{w/ } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

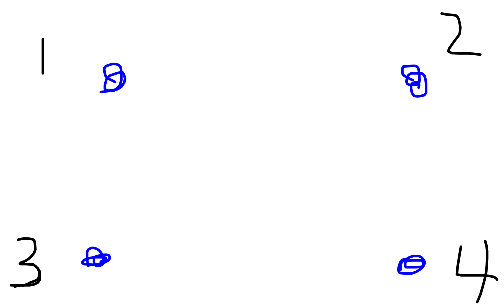
$$= -dU \quad (\text{dr's cancel})$$

So: $dT = -dU$ and $dE = d(T+U) = dT + dU = 0$ (Energy is conserved, as expected)

$$\boxed{E = T_1 + T_2 + U = \text{constant}}$$

Energy of Multiparticle Systems

For no external forces, have $T_{tot} = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$



$$U_{tot} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta}$$

ensures against double-counting

$$U_{tot} = \underbrace{U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}}_{\text{don't count } U\text{'s twice!}}$$

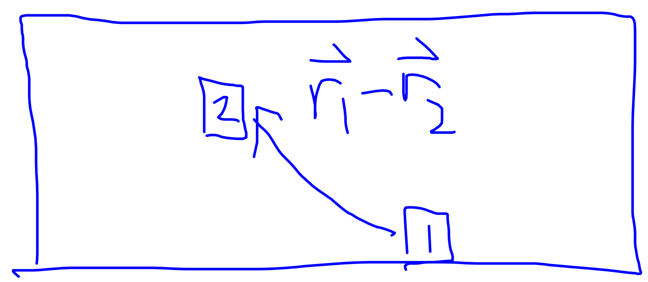
$$\boxed{E_{tot} = \sum_\alpha T_\alpha + \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta} = \text{const}}$$

pot. from ext. forces

internal potentials between particles

Adding external conservative force $\quad E_{tot} = \sum_\alpha T_\alpha + \sum_\alpha U_\alpha^{ext} + \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}^{int}$ is conserved

Rigid bodies



$\vec{r}_1 - \vec{r}_2$ is fixed $\Rightarrow$ U_{12} never changes

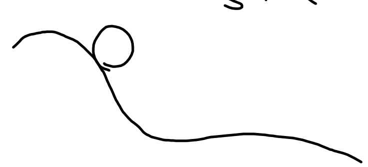
$$E_{tot} = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \sum_\alpha U_\alpha^{ext} + \underbrace{\sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}^{int}}_{\text{fixed, might as well be } \emptyset}$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \sum_\alpha U_\alpha^{ext}$$

$$= \frac{1}{2} M v^2 + U_{tot}^{ext}$$

$\boxed{\text{Rolling stone}}$

$I = \frac{2}{5} M R^2$



T