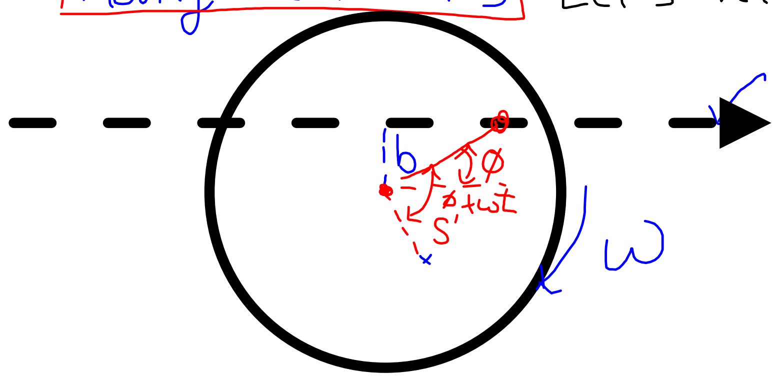
Lecture 02.2 - Physics 523

Moving Coordinates

Let's have a closer look at the hockey puck problem



puck trajectory in stationary observers frame

Cart. coord. $x = x_0 + vt$

$y = b$

Polar $x = vt = r\cos\phi$ ($x_0 = 0$)

$y = b = r\sin\phi$

$r^2 = b^2 + (vt)^2$

$\phi = \tan^{-1}\left(\frac{b}{vt}\right)$ restricted to I, II quadrants

What about for observer S'? → S & S' share origin

for S' $\phi' = \phi + \omega t$, $r' = r$

Puck trajectory in S' frame: $r' = \sqrt{b^2 + (vt)^2}$

$\phi' = \omega t + \tan^{-1}\left(\frac{b}{vt}\right)$

Angular Momentum (N- particles)

$$\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha \qquad \dot{\vec{L}} = \sum_{\alpha=1}^{N} \underbrace{\dot{\vec{r}}_\alpha \times \vec{p}_\alpha}_{0} + \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha = \vec{\tau}_{tot}$$

Now separate external & internal forces $\vec{F}_\alpha = \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{\,ext} \Rightarrow \dot{\vec{L}} = \underbrace{\sum_\alpha \sum_{\beta\neq\alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{\,ext} = \vec{\tau}^{\,ext}$

$$\frac{1}{2}\sum_\alpha \sum_{\beta\neq\alpha} \left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right)$$

$$= \frac{1}{2}\sum_\alpha \sum_{\beta\neq\alpha} \underbrace{\left(\vec{r}_\alpha - \vec{r}_\beta\right) \times \vec{F}_{\alpha\beta}}_{\text{0 for central forces}}$$

If $\vec{\tau}^{\,ext} = 0$, then $\vec{L}$ is conserved

Moments of Inertia

Can rewrite angular momentum in terms of angular velocity, ω

$L_z = I_z \omega_z$ ← moment about z-axis can be calculated

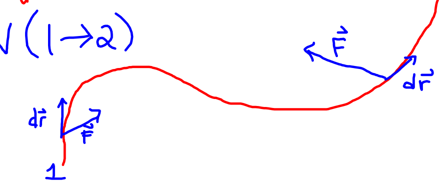
$I_i = \int \rho\, r_i^2\, dV$ (ρ ← density; r_i ← distance from axis)

Kinetic Energy / Work

Energy takes on a seemingly large number of forms (although all eventually traceable to kinetic + potential)

Kinetic $T = \frac{1}{2} m v^2$ for single particle $\frac{dT}{dt} = m\vec{v}\cdot\dot{\vec{v}} = \vec{v}\cdot\dot{\vec{p}} = \vec{v}\cdot\vec{F}$ or $dT = \underbrace{\vec{F}\cdot d\vec{r}}$ work done by $\vec{F}$ over $d\vec{r}$

For macroscopic displacements: $\Delta T = \underbrace{\int_1^2 \vec{F}\cdot d\vec{r}}_{\text{Line integral}} = \underbrace{W(1\to 2)}_{\text{work done to move it}}$



Note: for general forces, not the same for all paths

Potential Energy / Conservative forces

Recall that in E&M, $\Delta T = \int_1^2 \vec{F}\cdot d\vec{r}$ was independent of path (only endpoints mattered)

Electric force is conservative ($\exists$ associated potential energy)

General Conditions for conservativeness of forces $\rightarrow$ 1) F depends only on $\vec{r}$ (not $\vec{v}$, t, etc.) (gravity / electric forces)

$\vec{F} = q\vec{E}(r)$

2) W does not depend on path

Counterexamples: friction (violates 2)

air resistance (violates 2)

magnetic forces (violates 1)

So if force is conservative, can define potential energy $U(\vec{r}) \rightarrow$ function of position $E_{tot} = T + U$ is conserved

$$\boxed{U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'}$$

Nonsense for non-conservative forces

Note: $\vec{r}_0$ is arbitrary (equiv. the ϕ of U)

never measure U directly

Also have $\vec{F} = -\vec{\nabla}U(\vec{r})$ Can always write $\vec{F}$ as grad of function of position w/ conservative forces

$$W = \int_{\vec{r}_0}^{\vec{r}} \vec{\nabla}U \cdot d\vec{r} = U(\vec{r}) - U(\vec{r}_0)$$

Can have many conservative forces acting in concert $F_1, \dots, F_n$

If all forces are conservative, can define a U for each one $\rightarrow$ $\underbrace{E_{tot} = T + \overset{\text{gravity}}{U_1} + \overset{\text{electric potential E}}{U_2} + \cdots + U_n}_{\text{total is conserved}}$ ETC...

$\boxed{\text{Non-conservative forces}}$ No potential for that force $\rightarrow$ Separate conservative / non-cons. forces

(mechanical energy not conserved)

$$\vec{F} = \vec{F}_c + \vec{F}_{nc} \qquad \Delta T = W = \overset{-\Delta U}{W_c} + W_{nc} \qquad \Delta E_m = (\Delta T + \Delta U) = W_{nc}$$

$\uparrow$ there is U for this

energy lost to heat, for example

So we can write $U = -\int_{r_0}^{r} \vec{F}(r')\cdot d\vec{r}\,'$ $\rightarrow$ what about $\vec{F}$ in terms of U?

Note $W(\vec{r} \rightarrow \vec{r} + d\vec{r}) = \vec{F}(\vec{r})\cdot d\vec{r}$ for small displacements

$$= -dU = U(\vec{r}) - U(\vec{r}+d\vec{r}) = -\left[U(x+dx, y+dy, z+dz) - U(x,y,z)\right] = -\left[\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right]$$

$$= -\underbrace{\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)}_{\vec{\nabla} U}\cdot\underbrace{(\hat{x}dx + \hat{y}dy + \hat{z}dz)}_{d\vec{r}}$$

gradient of scalar function, U

$\boxed{\text{So we can identify } \vec{F} = -\vec{\nabla} U(\vec{r})}$

$\hookrightarrow$ possible for any conservative force

$\boxed{\text{The curl}}$ for conservative forces, path independence of $W \Rightarrow \boxed{\vec{\nabla} \times \vec{F} = 0}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{v}(\vec{r}) = \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

If $\vec{F} = -\vec{\nabla} U$, then $\vec{\nabla} \times \vec{F} = -\hat{x}\underbrace{\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z}U - \frac{\partial}{\partial z}\frac{\partial}{\partial y}U\right)}_{\emptyset} - \dots$ all others vanish also!

Curl of any gradient is vanishing

Recall we have 2 conditions for $\vec{F}$ to be conservative

1) $\vec{F}$ only depends on $\vec{r}$
2) $\vec{\nabla} \times \vec{F} = 0$

What if we only violate #1?

$\vec{F} = -\vec{\nabla} U(\vec{r}, t)$ ← time dep. potential energy

↳ $\vec{F}(\vec{r}, t)$

Still have $E = T + U$, but is E conserved?

$dE = dT + dU$

$$dT = \frac{dT}{dt} dt = \frac{d(\frac{1}{2} m \vec{v}^2)}{dt} dt = \vec{F} \cdot \vec{v}\, dt = \vec{F} d\vec{r}$$

$$dU = \frac{\partial U}{\partial x} dx + \dots + \frac{\partial U}{\partial t} dt = -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t} dt$$

$$\boxed{dE = \frac{\partial U}{\partial t} dt \neq 0}$$

So mechanical energy is not conserved in this case

Important distinctions ⇒ explicit vs. implicit time dependence

1) $\vec{F} = \vec{F}(\vec{r}(t))$ → time dependence only through evolution of $\vec{r}$ IMPLICIT
2) $\vec{F} = \vec{F}(\vec{r}(t), t)$ → force law itself evolves w/ t → EXPLICIT

e.g. $\frac{\partial}{\partial t} U(\vec{r}(t)) = 0$ (distinction between partial & total derivatives crucial!)

↑ implicit time dep.

for explicit time dep. $\frac{\partial}{\partial t} U(\vec{r}(t), t) \neq 0$

1D motion: All forces w/ $F = F(x)$ are conservative! Any 1D integral is path independent