Lecture 02.1 - Physics 523

The Rocket Problem

Rockets work by shedding mass → conservation of momentum ⇒ $\dot{\vec{P}} = 0$, but this is only on a closed system

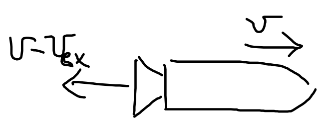
$\vec{P}_{tot} = \vec{P}_{gas} + \vec{P}_{rocket}$

these can individually change

$\dot{\vec{P}} = \dot{\vec{P}}_{gas} + \dot{\vec{P}}_{rocket} = 0$

@ time t $\vec{P}(t) = m(t)\,v(t)$

short time later: $\underbrace{m \to m + \overset{\text{negative}}{dm} \quad t \to t + \Delta t}_{\text{for the rocket}}$ $\qquad \underbrace{-dm, \text{ velocity } v - v_{ex}}_{\text{fuel}}$



$$P(t+dt) = (m+dm)(v+dv) - dm\,(v - v_{ex}) = mv + m\,dv + dm\,v_{ex}$$

If there is F_{ext} (eg gravity) then $dP = F_{ext}\,dt$

(HW was to solve this for $F_{ext} = -mg$)

Say $F_{ext} = 0 \Rightarrow \quad dP = m\,dv + dm\,v_{ex} \;\to\; \dfrac{dP}{dt} = m\dot{v} + \dot{m}\,v_{ex} = 0$

$m\dot{v} = \underbrace{-\dot{m}\,v_{ex}}_{}$ Like a force → "Thrust"

$$\frac{dv}{v_{ex}} = -\frac{dm}{m}$$

$$\frac{v - v_0}{v_{ex}} = -\ln\frac{m}{m_0} \qquad \boxed{v = v_0 + v_{ex}\ln\frac{m_0}{m}}$$

Center of Mass

N-particles → $\vec{R} = \dfrac{1}{M}\displaystyle\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha$

↑ total mass

2-particles $\quad \vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \simeq \vec{r}_1$ for $m_1 \gg m_2$

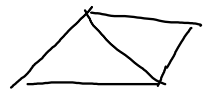
take time der. of CM eq. → $M\dot{\vec{R}} = \displaystyle\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} \vec{p}_\alpha = \vec{P}_{tot}$ total $\boxed{\vec{P} = M\dot{\vec{R}}}$

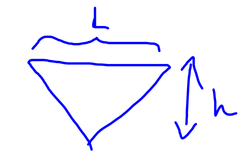
constant mass

↑ only need velocity of CM

Also $\quad M\ddot{\vec{R}} = \dot{\vec{P}}_{tot} = \vec{F}_{ext}$ ⇒ Whole system of particles has collective behavior of single particle at position $\vec{R}$.

CM for solid bodies $\vec{R} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\int \vec{r}\,\underbrace{\rho}_{\text{density}}\,dV$

CM of the pyramids  height h, base has sides of length L, density ρ



$\ell = \frac{L}{h} z$

$$dV = \ell^2 dz = \left(\frac{L}{h}\right)^2 z^2 dz \qquad R = \frac{1}{M}\int z\hat{z}\,\rho\left(\frac{L}{h}\right)^2 z^2 dz = \hat{z}\,\frac{\rho}{M}\left(\frac{L}{h}\right)^2\int_0^h z^3 dz = \hat{z}\,\frac{\rho L^2}{4 M}$$

now $M = \int \rho\, dV = \int \rho\left(\frac{L}{h}\right)^2 z^2 dz = \frac{\rho}{3}\frac{L^2}{h} \Rightarrow \boxed{R = \hat{z}\,\frac{2}{3}h}$ (but this is dist. from top)

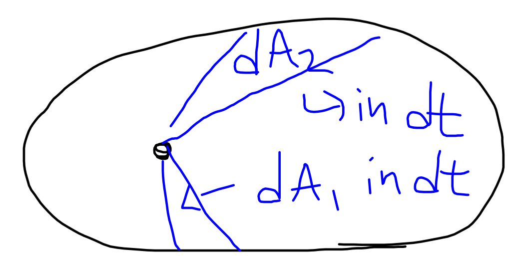
Angular Momentum

$\vec{\ell} \equiv \vec{r}\times\vec{p}$ (origin dependant!) $\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}} = \underbrace{\vec{v}\times\vec{p}}_{\phi \text{ since } \vec{p}\parallel\vec{v}} + \vec{r}\times\vec{F} = \vec{\tau}$ (Torque)

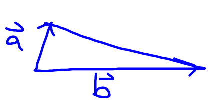
my old roommate's son's name

For "central" forces in closed systems $\rightarrow$ can choose $\mathcal{O}$ s.t. $\vec{\tau} = 0 \Rightarrow \vec{\ell}$ is conserved

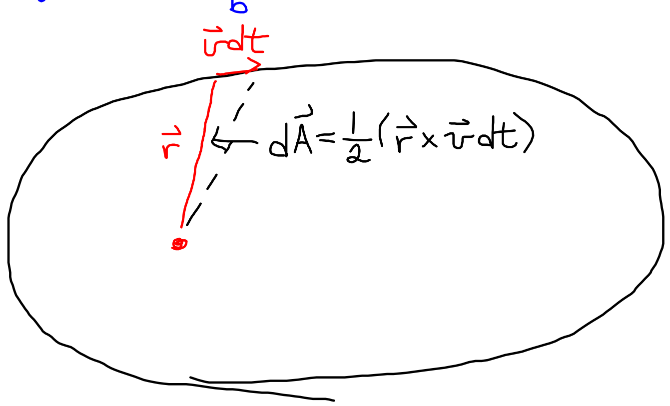
Kepler's 2nd Law Orbits subtend equal area per unit time:



$dA_1 = dA_2$

Triangles: 

$A = \frac{1}{2}|\vec{a}\times\vec{b}|$



$dA = \frac{1}{2}|\vec{r}\times\vec{v}\,dt| = \frac{dt}{2m}|\vec{r}\times\vec{p}| \Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2m}|\vec{\ell}| = \text{const}}$

★ Note: $\frac{1}{r^2}$ law never showed up! 2nd law true for any system w/ central Forces

Angular Momentum (N-particles)

$$\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha \qquad \dot{\vec{L}} = \sum_{\alpha=1}^{N} \underbrace{\dot{\vec{r}}_\alpha \times \vec{p}_\alpha}_{0} + \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha = \vec{\tau}_{tot}$$

Now separate external & internal forces $\vec{F}_\alpha = \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext} \Rightarrow \dot{\vec{L}} = \underbrace{\sum_\alpha \sum_{\beta\neq\alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = \vec{\tau}^{ext}$

$$\frac{1}{2}\sum_\alpha \sum_{\beta\neq\alpha} \left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right)$$

$$= \frac{1}{2}\sum_\alpha \sum_{\beta\neq\alpha} \underbrace{(\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}}_{\emptyset \text{ for central forces}}$$

If $\vec{\tau}^{ext} = 0$, then $\vec{L}$ is conserved

Moments of Inertia

Can rewrite angular momentum in terms of angular velocity, ω

moment about z-axis can be calculated

$$L_z = I_z \omega_z$$

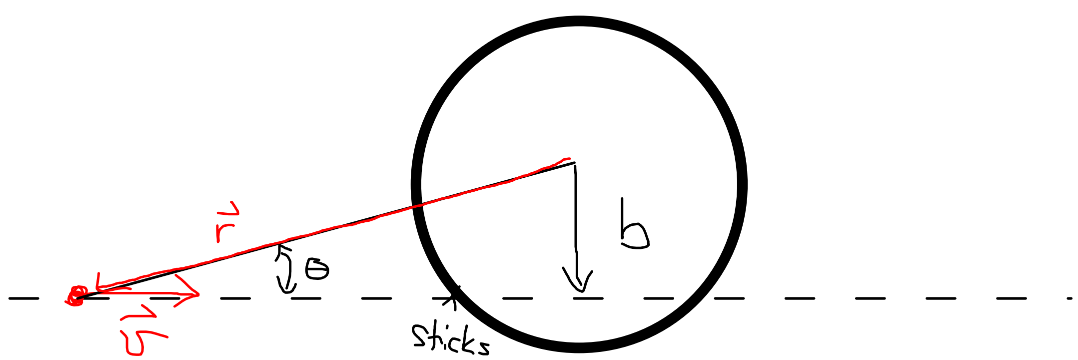
distance from axis

$$I_i = \int \rho r_i^2 \, dV$$

density

Example: Shooting spitballs @ the DJ

turntable has moment of inertia $I_z = \frac{1}{2} MR^2$



$$\vec{\ell}_0 = \vec{r} \times \vec{p} \qquad |\vec{\ell}_f| = I_{disk}\,\omega + I_{spitball}\,\omega = \left(\tfrac{1}{2} MR^2 + mR^2\right)\omega$$

$$|\vec{\ell}_0| = rmv\sin\theta$$

$$rmv\sin\theta = \left(m + \tfrac{1}{2}M\right)R^2\omega$$

$$mvb = \left(m + \tfrac{1}{2}M\right)R^2\omega$$

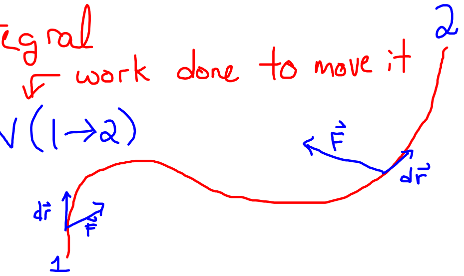
$$\boxed{\omega = \frac{vb}{\left(1 + \frac{1}{2}\frac{M}{m}\right)R^2}}$$

Kinetic Energy / Work

Energy takes on a seemingly large number of forms (although all eventually traceable to kinetic + potential)

Kinetic $T = \frac{1}{2} m v^2$ for single particle $\quad \frac{dT}{dt} = m\vec{v}\cdot\dot{\vec{v}} = \vec{v}\cdot\dot{\vec{p}} = \vec{v}\cdot\vec{F}$ or $dT = \underbrace{\vec{F}\cdot d\vec{r}}_{\text{work done by } \vec{F} \text{ over } d\vec{r}}$

For macroscopic displacements: $\Delta T = \overbrace{\int_1^2 \vec{F}\cdot d\vec{r}}^{\text{Line integral}} = \underbrace{W(1\to 2)}_{\text{work done to move it}}$



Note: for general forces, not the same for all paths

Potential Energy / Conservative forces

Recall that in E&M, $\Delta T = \int_1^2 \vec{F}\cdot d\vec{r}$ was independent of path (only endpoints mattered)

Electric force is conservative ($\exists$ associated potential energy)

General Conditions for conservativeness of forces $\rightarrow$ 1) F depends only on $\vec{r}$ (not $\vec{v}$, t, etc.) (gravity, electric forces)

$\vec{F} = q\vec{E}(r)$

2) W does not depend on path

Counterexamples: friction (violates 2)

air resistance (violates 2)

magnetic forces (violates 1)

So if force is conservative, can define potential energy $U(\vec{r}) \rightarrow$ function of position $\quad E_{tot} = T + U$ is conserved

$$\boxed{U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'}$$

Nonsense for non-conservative forces

Note: $\vec{r}_0$ is arbitrary (equiv. the ϕ of U)

never measure U directly

Also have $\vec{F} = -\vec{\nabla} U(\vec{r})$ $\quad$ Can always write $\vec{F}$ as grad of function of position w/ conservative forces

$$W = \int_{\vec{r}_0}^{\vec{r}} \vec{\nabla} U \cdot d\vec{r} = U(\vec{r}) - U(\vec{r}_0)$$