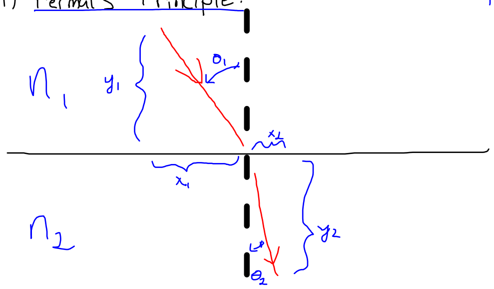
HW#2 Solutions

1) **Fermats Principle:**



$$T = \int \frac{ds}{v} = \frac{n_1}{c}\sqrt{x_1^2+y_1^2} + \frac{n_2}{c}\sqrt{x_2^2+y_2^2}$$

Call $x_1 + x_2 = \Delta x$ (Same for all paths)

$$= \frac{1}{c}\left[n_1\sqrt{x_1^2+y_1^2} + n_2\sqrt{(\Delta x - x_1)^2 + y_2^2}\right]$$

Want to minimize T wrt x_1: $\frac{\partial T}{\partial x_1} = \frac{1}{c}\left[n_1\frac{x_1}{l_1} - n_2\frac{(\Delta x - x_1)}{l_2}\right] = 0$

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$ Done!

2) **Center of mass** $\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i$

$$M^2R^2 = \left(\sum_i m_i \vec{r}_i\right)\cdot\left(\sum_j m_j \vec{r}_j\right) = \sum_{i,j} m_i m_j\, (\vec{r}_i\cdot\vec{r}_j)$$

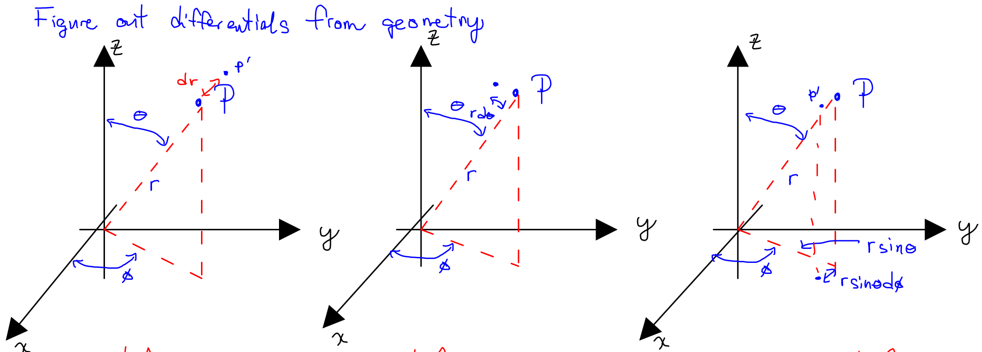
$\hookrightarrow -\frac{1}{2}(\vec{r}_i - \vec{r}_j)^2 + \frac{r_i^2}{2} + \frac{r_j^2}{2}$

$$= \sum_{i,j} m_i m_j\left[\frac{r_i^2}{2} + \frac{r_j^2}{2} - \frac{1}{2}(\vec{r}_i - \vec{r}_j)^2\right]$$

$$= \sum_i m_i \sum_j m_j \frac{r_i^2}{2} + (i \leftrightarrow j) - \frac{1}{2}\sum_{i,j} m_i m_j (\vec{r}_i - \vec{r}_j)^2$$

$$= \boxed{M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j(\vec{r}_i - \vec{r}_j)^2}$$

3) Figure out differentials from geometry



$dr\,\hat{r}$ $\quad$ $r\,d\theta\,\hat{\theta}$ $\quad$ $r\sin\theta\, d\phi\,\hat{\phi}$

I will show using chain rule + geometry: $d\vec{r} = d(r\hat{e}_r) = dr\,\hat{e}_r + d\theta\frac{\partial \hat{e}_r}{\partial\theta} + d\phi\frac{\partial\hat{e}_r}{\partial\phi}$

Considerp $\vec{r}$ for oint P to be unit vector $(r=1)$ then $d\vec{r} = d(\hat{e}_r) = dr\,\hat{r} + d\theta\,\hat{\theta} + \sin\theta\, d\phi\,\hat{\phi}$

$\Rightarrow \frac{\partial\hat{e}_r}{\partial\theta} = \hat{\theta} \quad \frac{\partial\hat{e}_\phi}{\partial\phi} = \sin\theta$

So $\boxed{d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}}$

Now directly differentiate x, y, z to get form of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$

$$dx\,\hat{e}_x = [dr\sin\theta\cos\phi + d\theta\, r\cos\theta\cos\phi - d\phi\, r\sin\theta\sin\phi]\,\hat{e}_x$$

$$dy\,\hat{e}_y = [dr\sin\theta\sin\phi + d\theta\, r\cos\theta\sin\phi + d\phi\, r\sin\theta\cos\phi]\,\hat{e}_y$$

$$dz\,\hat{e}_z = [dr\cos\theta - d\theta\, r\sin\theta]\,\hat{e}_z$$

So:
$$d\vec{r} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z = dr\Big[\overbrace{\sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z}^{\hat{e}_r}\Big]$$
$$+ r\,d\theta\Big[\overbrace{\cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z}^{\hat{e}_\theta}\Big]$$
$$+ r\sin\theta\, d\phi\Big[\underbrace{\cos\phi\,\hat{e}_y - \sin\phi\,\hat{e}_x}_{\hat{e}_\phi}\Big]$$

Part 2: from (4) in the HW, $\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\,\hat{e}_\phi \Rightarrow \frac{1}{2}m(\dot{\vec{r}})^2 = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right]$

$$\ddot{\vec{r}} = \frac{d}{dt}\left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\,\hat{e}_\phi\right] = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta + r\sin\theta\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\phi}\,\hat{e}_\phi$$
$$+ r\sin\theta\dot{\phi}\,\dot{\hat{e}}_\phi$$

Now we need $\dot{\hat{e}}_r$, $\dot{\hat{e}}_\theta$, and $\dot{\hat{e}}_\phi$

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta + \sin\theta\dot{\phi}\,\hat{e}_\phi$$
$$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r + \cos\theta\dot{\phi}\,\hat{e}_\phi$$
$$\dot{\hat{e}}_\phi = -\dot{\phi}[\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta]$$

Plug in & collect:

$$\boxed{\begin{aligned}\ddot{\vec{r}} = \hat{e}_r&\left[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right]\\ + \hat{e}_\theta&\left[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right]\\ + \hat{e}_\phi&\left[2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta\right]\end{aligned}}$$

4) Asteroids Angular momentum must be conserved $L_o = I_o \omega_o = I_f \omega_f$ $I_o = \frac{2}{5} M_o R_o^2$ $I_f = \frac{2}{5} M_f R_f^2 = \frac{8}{15}\pi\rho_o R_f^5$

L of dust is $\emptyset$

$$= \frac{2}{5}\left(\frac{4}{3}\pi R_o^3 \rho\right) R_o^2$$

$$= \frac{8}{15}\pi\rho R_o^5$$

So $\boxed{R_o^5 \omega_o = R_f^5 \omega_f}$ for $R_f = 2R_o$

$$R_o^5 \omega_o = (2R_o)^5 \omega_f \Rightarrow \boxed{\frac{\omega_f}{\omega_o} = \frac{1}{32}}$$