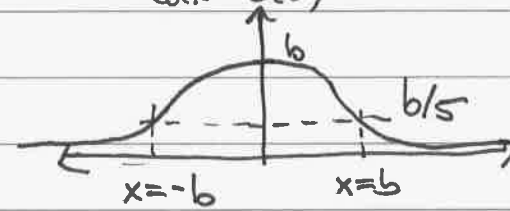
HW# 9 Solutions

① 7-13 in French $\quad y(x,t) = \dfrac{b^3}{b^2 + (2x-ut)^2}$

a) Sketch $y(x,t=0) = \dfrac{b^3}{b^2 + (2x)^2}$



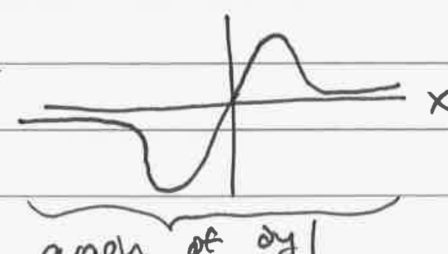
b) To obtain the wave speed, we put $y(x,t)$ into the form $f(x \pm vt)$:

$$y(x,t) = \frac{b^3}{b^2+(2x-ut)^2} = \frac{b^3}{b^2 + 4\underbrace{\left(x - \frac{u}{2}t\right)}_{\sim\, x - vt}^2}$$

→ So $\boxed{v = u/2}$, and the pulse moves in the $\boxed{+x\text{-dir.}}$

c) $\dfrac{\partial y}{\partial t} = \dfrac{+\, b^3(2x-ut)u}{\left[b^2 + (2x-ut)^2\right]^2}$ and @ $t=0$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{2\, x b^3 u}{(b^2+4x^2)^2}$$



graph of $\left.\frac{\partial y}{\partial t}\right|_{t=0}$

→ During a short time δt, points to the right of the origin move up, while those to the left go down. The net effect is to move the pulse to the right while maintaining its shape.

② 7-18 in French

Phase / group velocity of waterwaves

$$v_p = \left(\frac{2\pi S}{\rho\lambda}\right)^{1/2} \quad \text{w/ } S \equiv \text{surface tension}, \quad \rho \equiv \text{mass density}$$

a) $$v_g = v_p + k\frac{dv_p}{dk} = \left(\frac{2\pi S k}{\rho}\right)^{1/2} + k\left(\frac{2\pi S}{\rho}\right)^{1/2}\frac{1}{2}\frac{1}{\sqrt{k}}$$

$k = \frac{1}{\lambda}$

$$= \frac{3}{2}\left(\frac{2\pi S k}{\rho}\right)^{1/2} = \frac{3}{2} v_p$$

so $$\boxed{v_g = \frac{3}{2} v_p}$$

b) The net disturbance will progress with velocity v_g, which is faster than the motion of the ripples themselves, which move with velocity v_{phase}. If you were to keep pace with the group of ripples, the ripples would appear to be moving backwards.

c) $$\lambda_{crest} = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^{-1} = \left(\frac{1}{.99\,\text{cm}} - \frac{1}{1.01\,\text{cm}}\right)^{-1} = 50\,\text{cm}$$

$$\Rightarrow \boxed{\lambda_{crest} = 50\,\text{cm}}$$

This is due to a beat phenomenon:

$$\cos\left(\frac{2\pi}{\lambda_1}x\right) + \cos\left(\frac{2\pi}{\lambda_2}x\right) \propto \cos\left[\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x\right]$$

③ 7-24 in French Longitudinal wave in rod

$$\xi = \xi_0 \cos\left[2\pi k(x - vt)\right]$$

mass density ρ, areas S, modulus Y

The KE of one segment is $\frac{1}{2}\, dm \left(\frac{\partial \xi}{\partial t}\right)^2 = \frac{1}{2}\rho S\,\Delta x \left(\frac{\partial \xi}{\partial t}\right)^2$

$$\Rightarrow \boxed{\frac{dKE}{dx} = \frac{1}{2}(\rho S)\left(\frac{\partial \xi}{\partial t}\right)^2}$$

The force on a segment of stressed rod is

$$F = -\frac{SY}{l_0}x \Rightarrow U = \frac{1}{2}\frac{SY}{l_0}x^2$$

now $x \sim \frac{\partial \xi}{\partial x}\Delta x$, and $l_0 = \Delta x$

$$\Rightarrow dU = \frac{1}{2}\frac{SY}{\Delta x}\left(\frac{\partial \xi}{\partial x}\Delta x\right)^2 \Rightarrow \boxed{\frac{dU}{dx} = \frac{1}{2}SY\left(\frac{\partial \xi}{\partial x}\right)^2}$$

$$U_\lambda = \int_0^\lambda dx\,\frac{dU}{dx} = \int_0^\lambda dx\,\frac{1}{2}SY(2\pi k)^2\xi_0^2\sin^2\left[2\pi k(x - vt)\right]$$

$$= \frac{1}{2}SY\lambda\,\underbrace{(2\pi k\xi_0)^2}_{\frac{v_{max}^2}{v_{wave}^2}}\,\frac{1}{2} = \frac{1}{4}\frac{SY\lambda u_0^2}{\underset{\uparrow\, Y/\rho}{v_{wave}^2}}$$

$$\boxed{U_\lambda = \frac{1}{4}\rho S\lambda u_0^2}$$

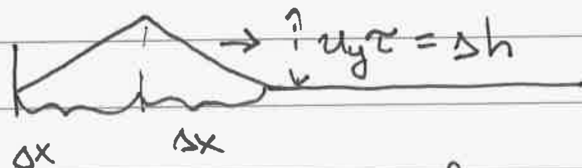
$$K_\lambda = \int_0^\lambda dx\,\frac{dKE}{dx} = \int_0^\lambda dx\,\frac{1}{2}(\rho S)\underbrace{\xi_0^2(2\pi k v)^2}_{u_0^2}\sin^2\left[2\pi k(x - vt)\right]$$

$$\boxed{K_\lambda = \frac{1}{4}\rho S\lambda u_0^2}$$

④ 7-23 in French

The total KE is $\left[\frac{1}{2}\mu\Delta x(u_y^2)\right]\times 2$

where $\Delta x = v_{wave}\cdot\tau$



$$\Rightarrow \boxed{KE = \mu v\tau(u_y)^2}$$

Total PE is $\frac{1}{2}\int dx\, T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}T\left(\frac{u_y\cancel{\tau}}{v_{wave}\cancel{\tau}}\right)^2\times 2\Delta x$

$$= \cancel{\tfrac{1}{2}}\cancel{T}\left(\frac{\mu}{\cancel{T}}\right)(u_y)^2\times\cancel{2}\times v\tau$$

$$\Rightarrow \boxed{PE = \mu v\tau(u_y)^2\ (=KE)}$$

$$E_{tot} = 2\mu v\tau(u_y^2) = 2\mu v u_y\,\Delta h$$
$$= 2T\left(\frac{u_y}{v}\right)\Delta h = 2T\left|\frac{\partial y}{\partial x}\right|\Delta h$$

$\Rightarrow$ Note that $T\left|\frac{\partial y}{\partial x}\right|$ is the magnitude of the restorative force, and Δh is the distance over which it is applied (first upwards, and then downwards, hence the overall factor of 2

$$\Rightarrow \boxed{E_{tot} = -\int F\cdot dx = T\left|\frac{\partial y}{\partial x}\right|(2\Delta h) = W}$$

work done on system

⑤ <u>8-1 in French</u>

Junction of 2 strings, μ_1 + μ_2, traveling wave incident on boundary

→ Want <u>amplitude</u> of reflected / transmitted waves

→ Assume wave starts in region 1, with $\mu = \mu_1$

Initial pulse $f_1(x - v_1 t)$ or $f_1(t - x/v_1)$

@ boundary $\underbrace{g_1(t)}_{\text{reflected}} = \dfrac{v_2 - v_1}{v_2 + v_1} f_1(t)$

$$\Rightarrow A_{refl} = \frac{v_2 - v_1}{v_2 + v_1} A_{initial} = \frac{\frac{1}{\sqrt{\mu_2}} - \frac{1}{\sqrt{\mu_1}}}{\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_2}}} A_{initial}$$

$$\Rightarrow \boxed{A_{refl} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A_{init}}$$

~~Atram~~ Similarly, $f_2(t) = \dfrac{2v_2}{v_2 + v_1} f_1(t)$

$$\Rightarrow A_{trans} = \frac{2v_2}{v_2 + v_1} A_{init} = \frac{2/\sqrt{\mu_2}}{1/\sqrt{\mu_2} + 1/\sqrt{\mu_1}} A_{init} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}} A_{init}$$

$$\Rightarrow \boxed{A_{trans} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}} A_{init}}$$

μ_2/μ_1

μ_2/μ_1 →	0	1/4	1	4	∞
A_t/A_i	2	4/3	1	2/3	0
A_r/A_i	1	1/3	0	-1/3	-1