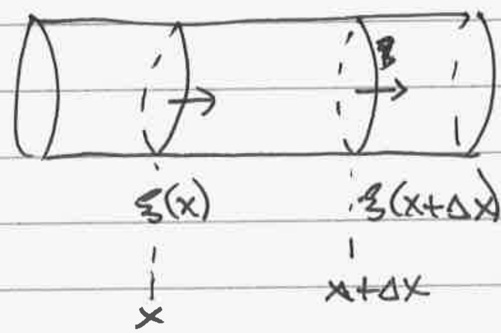
HW# 7 Solutions

① 6-7 in French Have $K = -V\frac{dp}{dV} \rightarrow \Delta p = -\frac{K}{V}\Delta V$



$$\Delta p = -\frac{K}{\Delta x}\left(\xi(x+\Delta x) - \xi(x)\right)$$

$$\rightarrow -K\frac{\partial \xi}{\partial x} \quad \text{as } \Delta x \rightarrow 0$$

total force on air between x and $x+\Delta x$:

$$F = A\Delta p_x - A\Delta p_{x+\Delta x} = A\left[-K\left(\left.\frac{\partial \xi}{\partial x}\right|_x - \left.\frac{\partial \xi}{\partial x}\right|_{x+\Delta x}\right)\right]$$

$$\cong AK\Delta x\frac{\partial^2 \xi}{\partial x^2}$$

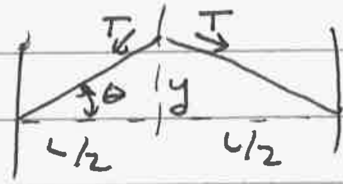
Newton's 2nd: $= \Delta m\frac{\partial^2 \xi}{\partial t^2}$ (ma)

$(\Delta m = \rho A\Delta x)$

$$\Rightarrow \cancel{A}K\cancel{\Delta x}\frac{\partial^2 \xi}{\partial x^2} = \rho\cancel{A}\cancel{\Delta x}\frac{\partial^2 \xi}{\partial t^2} \Rightarrow \boxed{\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{K}\frac{\partial^2 \xi}{\partial t^2}}$$

② 6-12 in French

a) This is easiest to do by calculating work done. This configuration can be achieved by pushing only on center of string:



$F = -2T\sin\theta \cong -2T\frac{y}{(L/2)}$

$\Rightarrow F \cong -4Ty$

$U = -\int_0^h dy\, F_y = 4T\int_0^h y = 2Th^2$

$\boxed{E_{tot} = \cancel{2}Th^2}$ (all energy in U initially, as $V=0$)

b) Recall that periods of normal modes are

$T_n = \frac{T_1}{n}$

Clearly, fourier component of lowest mode is non zero

$2\int_0^{L/2}\left(\frac{2h}{L}\right)x\sin\left(\frac{\pi x}{L}\right) \neq 0$

So system repeats with period $T = T_1 = \frac{2\pi}{\omega_1}$

$\omega_1 = \frac{\pi v}{L} \Rightarrow \boxed{T = \frac{2L}{v}}$

③ <u>6-14 In French</u> Finding fourier series

a) $y(x) = Ax(L-x)$ → parabola w/ peak @ $x = L/2$

$$B_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L}A\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2AL^2(2-2\cos\pi n)}{\pi^3 n^3} = \frac{4AL^2(1-(-1)^n)}{\pi^3 n^3}$$

$$= \begin{cases} \dfrac{8AL^2}{\pi^3 n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

b) $y(x) = A\sin\left(\frac{\pi x}{L}\right)$ $\quad B_n = \frac{2}{L}A\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \begin{cases} 0 & \text{if } n \neq 1 \text{ by orthogonality} \\ \dfrac{A}{L} & \text{if } n = 1 \end{cases}$$

since $\int_0^L \sin^2\left(\frac{\pi x}{L}\right)dx = \frac{1}{2}$

c) $y(x) = \begin{cases} A\sin\left(\frac{2\pi x}{L}\right) & 0 \leq x \leq L/2 \\ 0 & L/2 \leq x \leq L \end{cases}$

$$B_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\int_0^{L/2} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \frac{2}{L}\int_{L/2} \phi\, dx$$

$$\boxed{B_n = \begin{cases} \frac{1}{2} & n=2 \text{ (special case) or 0 in n is even otherwise} \\ -\dfrac{4\sin(\pi n/2)}{\pi(n^2-4)} & \text{otherwise } (n \text{ is odd}) \end{cases}}$$

(4) <u>6-15 in French</u> Here, we plug in time dependence of normal modes

a) $y(x,0) = Ax(L-x) \quad (\partial y/\partial t)_{t=0} = 0$

First, fourier decompose $y(x,0) = \sum_{\substack{n=1 \\ (n\text{-odd})}}^{\infty} \frac{8AL^2}{\pi^3 n^3} \sin\left(\frac{n\pi x}{L}\right)$

from 6-14, part a.

This is composition in terms of <u>normal modes</u>

→ time dependence of each is $\cos(\omega_n t - \delta_n)$

since $(\partial y/\partial t)_{t=0} = 0$, $\delta_n = 0$ for all n

$$\Rightarrow \boxed{y(x,t) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{8AL^2}{\pi^3 n^3} \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t) \quad \left(\text{with } \omega_n = \frac{n\pi v}{L}\right)}$$

b) now we have $\left(\frac{\partial y}{\partial t}\right) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{8AL^2}{\pi^3 n^3} \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$

by the same reasoning → integrate over time to get position

$$\int dt \cos(\omega_n t) = \frac{\sin \omega_n t}{\omega_n} = \frac{(\sin \omega_n t) L}{n\pi v}$$

$$\Rightarrow y(x,t) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{8AL^2}{\pi^3 n^3} \left(\frac{L}{n\pi v}\right) \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t)$$

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{8AL^3}{\pi^4 n^4 v} \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t)}$$