HW#6 Solutions

① 5-14 in French $A_{pn} = C_n \sin\left(\frac{n\pi}{N+1} p\right)$

for $N=3$, question asks for relative amplitudes
$\Rightarrow$ can take $C_n = 1$ for $n = 1,2,3$

Array:

$$\begin{array}{c} \\ n=1 \\ n=2 \\ n=3 \end{array} \begin{array}{c} \begin{array}{ccc} p=1 & p=2 & p=3 \end{array} \\ \begin{pmatrix} 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 1 & 0 & -1 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \end{pmatrix} \end{array}$$

Note: so long as ratios of amplitudes are the same, answer is correct

② 6-1 in French String with mass .01 kg, $L = 2.5\text{m}$
$T = 10\text{N}$

a) $\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{(M/L)}} = n\pi\sqrt{\frac{T}{ML}}$

Fundamental is for $n = 1 \rightarrow \omega_1 = \pi\sqrt{\frac{T}{ML}} = \pi\sqrt{\frac{10\text{N}}{(.01\text{kg})(2.5\text{m})}}$

$$\boxed{\omega_1 = 20\pi \text{ rad/s}}$$

b) If the string is touched at 0.5m, or at $\frac{1}{5}L$, only modes with nodes at $x = .05\text{m}$ persist. Like $L_{new} = L/5$, $\left(\omega_n = \frac{5n\pi}{L}\sqrt{\frac{T}{(M/L)}}\right)$

$\Rightarrow$ Persisting modes are $\boxed{n = 5, 10, 15, \ldots}$

③ 6-2 in French String with length L, mass M, tension T

Normal mode frequencies are:

$$\boxed{\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} = n\pi\sqrt{\frac{T}{ML}}}$$

for finite N in the discrete mass case, we had: $\omega^2 = 4\omega_0^2 \sin^2\left(\frac{n\pi}{2(N+1)}\right)$ with $\omega_0^2 = \frac{T}{ml}$

$$\omega_n^2 = \frac{4T}{\left(\frac{M}{3}\right)\left(\frac{L}{4}\right)} \sin^2\left(\frac{n\pi}{2(3+1)}\right) = \frac{T}{ML} \cdot 48 \sin^2\left(\frac{n\pi}{8}\right)$$

So
$$\boxed{\begin{aligned} \frac{\omega_n^{\text{string}}}{\pi\sqrt{\frac{T}{ML}}} &= 1, 2, 3, \ldots \\ \frac{\omega_n^{\text{3 masses}}}{\pi\sqrt{\frac{T}{ML}}} &= \frac{\sqrt{48}}{\pi} \cdot \left(\sin\left(\frac{\pi}{8}\right), \sin\left(\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{8}\right)\right) \\ &= 0.84, 1.56, 2.03 \end{aligned}}$$

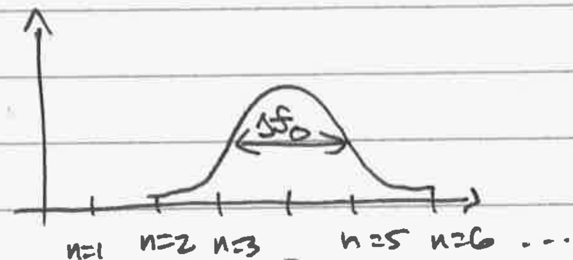
Not terribly close for N so small!

④ 6-10 in French Laser → produce light by exciting normal modes of cavity

a) Since the two mirrors act like rigid walls, normal mode spectrum is like that of string fixed at both ends: $\boxed{\omega_n = \frac{n\pi v}{L} = \frac{n\pi c}{L}}$

b) We now pump light into cavity with a spread of frequencies $f_0 \pm \frac{\Delta f_0}{2}$

4b – cont'd in general, laying the normal mode frequencies on top of the excitation signal:



$f_n = \frac{nc}{2L}$ many normal modes lie in interval $f_0 - \frac{\Delta f_0}{2}$ to $f_0 + \frac{\Delta f_0}{2}$

(1) $L = 1.5\,m$, $f_0 = 5 \times 10^{14}\,Hz$, $\Delta f_0 = 2 \times 10^9\,Hz$

now $f_1 = \frac{c}{2L} = \frac{3 \times 10^8\,m/s}{2\,(2.5\,m)} = 6 \times 10^7\,Hz$

which is also the spacing between normal modes. Thus, # of normal modes excited is $\frac{\Delta f_0}{f_1} = \frac{2 \times 10^9\,Hz}{6 \times 10^7\,Hz} \approx 3.3 \times 10^1$ modes

$\boxed{33 \text{ modes}}$

(2) To get only 1 mode, $\cancel{\frac{\Delta f_0}{f_1}}$ require spacing to be equal to spread

$\frac{\Delta f_0}{f_1} = \Delta f_0 \left(\frac{2L}{c}\right) = 1 \Rightarrow L = \frac{c}{2\Delta f_0} = \frac{3 \times 10^8\,m/s}{2(2 \times 10^9\,Hz)}$

$\boxed{L = .075\,m}$ → table-top laser cavity!

Note central value f_0 played no role!

⑤ 6-11 In French

a) The total energy of a string in a normal mode n can be found by considering the integral over the KE's of small segments of string:

$$dKE = \frac{1}{2}\,dm\,v^2 = \frac{1}{2}(\mu\,dx)\left(\frac{\partial y(x,t)}{\partial t}\right)^2$$

In the n'th normal mode, $y_n = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$

$$\Rightarrow dKE = \frac{1}{2}(\mu dx)\left[-A_n\omega_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t)\right]^2$$

$$KE = \int dKE = \frac{1}{2}\mu A_n^2 \omega_n^2 \sin^2(\omega_n t)\underbrace{\int_{x=0}^{x=L} \sin^2\left(\frac{n\pi x}{L}\right)dx}_{L/2}$$

$$\boxed{E_{tot} = KE_{max} = \frac{A_n^2}{2}\cancel{\mu}\left(\frac{n^2\pi^2 T}{L^2\,\cancel{\mu}}\right)\left(\frac{L}{2}\right) = \frac{A_n^2}{4}\,\frac{n^2\pi^2 T}{L}}$$

b) Write the velocities as $\frac{\partial y}{\partial t} = \frac{\partial y_1}{\partial t} + \frac{\partial y_3}{\partial t}$

1st + 3rd normal mode

Now $d\cancel{KE} = \frac{1}{2}\mu dx\left(\frac{\partial y_1}{\partial t} + \frac{\partial y_3}{\partial t}\right)^2 = \frac{1}{2}\mu dx\left[\left(\frac{\partial y_1}{\partial t}\right)^2 + \left(\frac{\partial y_3}{\partial t}\right)^2 + 2\frac{\partial y_1}{\partial t}\frac{\partial y_3}{\partial t}\right]$

cross term

this will give sum over E's of each normal mode

The cross term:

$$\int dKE^{cross} \propto \int_0^L dx\,\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right) = 0\,!$$ Cross-term vanishes

Q11 cont'd Note that the phase shift does not matter. The energy is just the sum over the partitioning of energies into each mode, so can just consider max KE_1 and max KE_3 independently

$$\boxed{E_{tot} = E_1 + E_3 = \frac{A_1^2}{4} \cdot \frac{\pi^2 T}{L} + \frac{A_3^2}{4} \cdot \frac{9\pi^2 T}{L}}$$