# HW#5 Solutions

① **Problem 5-2** 2 Identical Pendula, coupled by spring (just like demo) with $l = 0.4\,m$, $g = 9.8\,m/s^2$ $T_{clamp} = 1.25\,s = \frac{2\pi}{\omega_c} = 2\pi\sqrt{\frac{m}{k}}$

$\omega_c$ in lectures

a) We can use symmetry or brute force to obtain normal modes. Symmetry is more elegant, but brute force more generally applicable. We will use brute force for 5-10, so let's hold off and use symmetry for this one.

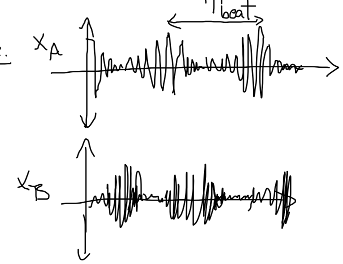
Symmetry: the two modes will have $x_A = x_B$ (symmetric) and $x_A = -x_B$ (anti-symmetric)

Equations of motion:
$$F_A = -m\omega_0^2 x_A - k(x_A - x_B) = m\ddot{x}_A$$
$$F_B = -m\omega_0^2 x_B - k(x_B - x_A) = m\ddot{x}_B$$
$$m(\ddot{x}_A + \ddot{x}_B) = -m\omega_0^2(x_A + x_B) \quad\Rightarrow\quad (x_A + x_B) = A_+\cos(\omega_0 t + \delta_+)$$
$$m(\ddot{x}_A - \ddot{x}_B) = -(m\omega_0^2 + 2k)(x_A - x_B) \quad\Rightarrow\quad (x_A - x_B) = A_-\cos\left(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \delta_-\right)$$

Periods are
$$\boxed{T_+ = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{.4\,m}{9.8\,m/s^2}} = 1.27\,s}$$
$$\boxed{T_- = \frac{2\pi}{(\omega_0^2 + 2\omega_c^2)^{1/2}} = \frac{2\pi}{\left[\left(\frac{.4\,m}{9.8\,m/s^2}\right) + 2\left(\frac{2\pi}{1.25\,s}\right)^2\right]^{1/2}} = 0.88\,s}$$

b) They are asking, for a general excitation, when the amplitude of one of the degrees of freedom undergoes 1 cycle (the beat period)

i.e.




Recall the sum of two sinusoids w/ 2 frequencies (i.e. our 2 normal modes) creates a beat phenomenon: $\cos(\omega t) + \cos(\omega' t) \sim \cos(\bar{\omega} t)\cos\left(\frac{1}{2}(\omega - \omega')t\right)$

$$\boxed{\text{time between amplitude max is } \frac{2\pi}{\omega_{beat}} = \frac{2\pi}{\omega - \omega'} = \frac{1}{\frac{1}{T_-} - \frac{1}{T_+}} = 2.9 \text{ seconds}}$$

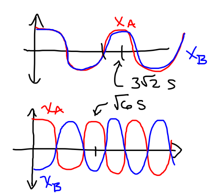
② **5-6 in French** Clamping either mass, $F_{free} = -2kx = m\ddot{x} \Rightarrow \ddot{x} = -\omega_0^2 x = -\frac{2k}{m}x$ with $\omega_0 = \frac{2\pi}{T_{free}} = \frac{2}{3}\pi\, s^{-1}$

a) The equations of motion are
$$F_A = -kx_A - k(x_A - x_B) = m\ddot{x}_A$$
$$F_B = -kx_B - k(x_B - x_A) = m\ddot{x}_B$$
$$\ddot{x}_A + \overset{2k/m}{\omega_0^2}\left(x_A - \frac{x_B}{2}\right) = 0$$
$$\ddot{x}_B + \omega_0^2\left(x_B - \frac{1}{2}x_A\right) = 0$$
$$(\ddot{x}_A + \ddot{x}_B) + \frac{\omega_0^2}{2}(x_A + x_B) = 0 \qquad \omega_+^2 = \frac{\omega_0^2}{2}$$
$$(\ddot{x}_A - \ddot{x}_B) + \frac{3\omega_0^2}{2}(x_A - x_B) = 0 \qquad \omega_-^2 = \frac{3\omega_0^2}{2}$$

Periods are
$$\boxed{\begin{aligned} T_+ &= \frac{2\pi}{\omega_+} = \frac{2\pi}{\omega_0}\sqrt{2} = T_{free}\sqrt{2} = 3\sqrt{2}\,s \\ T_- &= \frac{2\pi}{\omega_-} = \frac{2\pi}{\omega_0}\sqrt{\frac{2}{3}} = T_{free}\sqrt{\frac{2}{3}} = \sqrt{6}\,s \end{aligned}}$$

Symmetric mode:

Anti-Sym




b) + c) Now, at $t=0$, $x_A = 0$, $x_B = 5\,cm$; $\dot{x}_A = 0$, $\dot{x}_B = 0$

$$x_A + x_B = (5\,cm)\cos(\omega_+ t)$$
$$x_A - x_B = (-5\,cm)\cos(\omega_- t)$$

$$x_A = \frac{5\,cm}{2}\left[\cos(\omega_+ t) - \cos(\omega_- t)\right] \qquad x_B = \frac{5\,cm}{2}\left[\cos(\omega_+ t) + \cos(\omega_- t)\right]$$

$$T_{cycle} = \frac{2\pi}{\omega_+ - \omega_-} = \left|\frac{1}{\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{6}}}\right| = \frac{3\sqrt{2}}{\sqrt{3} - 1} = 5.8\,s$$

Note $\frac{\omega_+}{\omega_-} = \sqrt{3}$ which is irrational $\Rightarrow$ motion of 2 d.o.f. system never exactly repeats!

③ 5-9 in French See lecture 7.2! Note This is the principle by which microwave ovens work. The microwaves are tuned to resonant frequencies of water molecules → act as driving force on vibrations which dissipate as heat!

④ 5-10 in French here, $U = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1-x_2)^2 \Rightarrow$

$$F_1 = -k(x_1 + x_1 - x_2) = -k(2x_1 - x_2) = m\ddot{x}_1 \qquad \ddot{x}_1 + \frac{k}{m}(2x_1 - x_2) = 0$$
$$F_2 = -k(-1)(x_1 - x_2) = k(x_1 - x_2) = m\ddot{x}_2 \qquad \ddot{x}_2 + \frac{k}{m}(x_2 - x_1) = 0$$

For normal modes $\left.\begin{array}{l} x_1 = A_1\cos\omega t \\ x_2 = A_2\cos\omega t\end{array}\right\}$ Plug in

$$[-A_1\omega^2 + \omega_0^2(2A_1 - A_2)] = 0 \rightarrow \frac{A_1}{A_2}\left(2-\left(\frac{\omega}{\omega_0}\right)^2\right) - 1 = 0 \qquad \left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\left(2-\left(\frac{\omega}{\omega_0}\right)^2\right) - 1 = 0$$
$$[-A_2\omega^2 + \omega_0^2(A_2 - A_1)] = 0 \rightarrow \frac{A_1}{A_2} = 1-\left(\frac{\omega}{\omega_0}\right)^2 \qquad \left(\frac{\omega}{\omega_0}\right)^4 - 3\left(\frac{\omega}{\omega_0}\right)^2 + 1 = 0$$

Frequencies:

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{3 \pm \sqrt{9-4}}{2} \Rightarrow \boxed{\frac{\omega_+}{\omega_-} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}}} = \sqrt{\frac{(1+\sqrt{5})^2/2}{(1-\sqrt{5})^2/2}} = \frac{1+\sqrt{5}}{1-\sqrt{5}}$$

$A_1/A_2$:

$$\boxed{\frac{A_1}{A_2} = 1-\left(\frac{\omega}{\omega_0}\right)^2 = 1 - \frac{3\pm\sqrt{5}}{2} = \frac{\pm\sqrt{5}-1}{2}}$$

⑤ The equations of motion are

$$F_A = -k(x_A - X_0\cos\omega t) - k(x_A - x_B) = m\ddot{x}_A \qquad \ddot{x}_A + \omega_0^2(2x_A - x_B) = X_0\cos\omega t$$
$$F_B = -k(x_B - x_A) - kx_B = m\ddot{x}_B \qquad \ddot{x}_B + \omega_0^2(2x_B - x_A) = 0$$

Response will be $x_A = A\cos\omega t$, $x_B = B\cos\omega t$

Plug in $[-A\omega^2 + \omega_0^2(2A - B)] = X_0$

$$[-B\omega^2 + \omega_0^2(2B - A)] = 0 \rightarrow \frac{A}{B} = 2 - \left(\frac{\omega}{\omega_0}\right)^2$$

1st eq is then

$$\omega_0^2 B\left[-\frac{\omega^2}{\omega_0^2}\left(2-\left(\frac{\omega}{\omega_0}\right)^2\right) + \left(2\left(2-\left(\frac{\omega}{\omega_0}\right)^2\right) - 1\right)\right] = \omega_0^2 B\Bigg[\underbrace{\left(\frac{\omega}{\omega_0}\right)^4 - 4\left(\frac{\omega}{\omega_0}\right)^2 + 3}_{\left(\frac{\omega^2-\omega_+^2}{\omega_0^2}\right)\left(\frac{\omega^2-\omega_-^2}{\omega_0^2}\right)}\Bigg] = X_0$$

$$\left(\frac{\omega_+}{\omega_0}\right)^2 = 3, \quad \left(\frac{\omega_-}{\omega_0}\right)^2 = 1$$

Not really clear what they want for graphs of displacements... depends on parameters..., mainly $\omega$