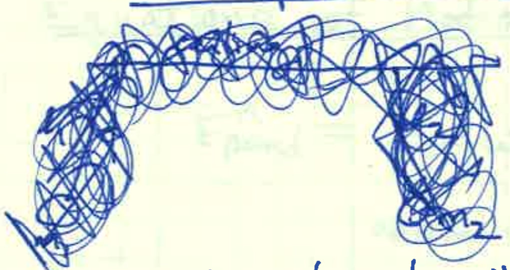
1

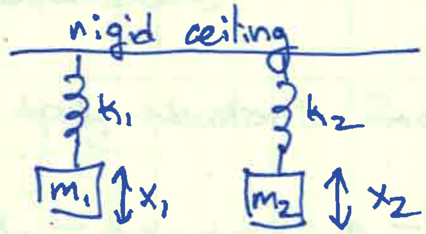
"Coupled" Oscillators

Lecture 5.2 We have studied ~~the~~ various incarnations of oscillators with one degree of freedom SHO, damped SHO, damped driven SHO

If systems are uncoupled (they do not influence each other) then analysis is "simple" → just do the SHO (or others) N times, once for every degree of freedom.

Uncoupled SHO's :





$\frac{k_1}{m_1}$ $\qquad$ $\frac{k_2}{m_2}$

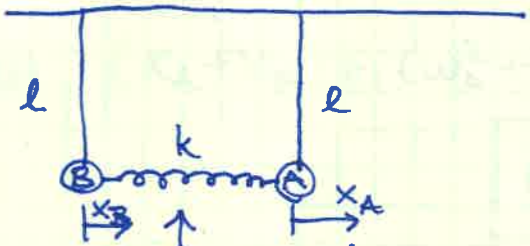
$\ddot{x}_1 = -\omega_1^2 x_1$ $\qquad$ $\ddot{x}_2 = -\omega_2^2 x_2$

degrees of freedom don't "talk" to each other, don't inluence each others motion

In almost all cases, there is crosstalk between degrees of freedom

$\boxed{\text{Demo video of locking metronomes}}$

Consider now adding a coupling between 2 identical pendula:



spring facilitates crosstalk
"couples" the oscillators

I. Demo: energy shifts between one oscillator and the next

(excite A, drives B through spring, energy transfer starts)

Equations of Motion Apply Newton's 2nd law

$$F^A_{pend} = \underset{\substack{\uparrow \\ \text{natural freq. of pendulum}}}{-m\omega_0^2 x_A} - kx_A + kx_B = -m\omega_0^2 x_A - k(x_A - x_B) = m\ddot{x}_A$$

$$F^B_{pend} = -m\omega_0^2 x_B - kx_B + kx_A = -m\omega_0^2 x_B - k(x_B - x_A) = m\ddot{x}_B$$

2 coupled 2nd order linear homogeneous diff. equations

$$\boxed{\begin{aligned} \ddot{x}_A + (\omega_0^2 + \omega_c^2)\, x_A - \omega_c^2 x_B &= 0 \\ \ddot{x}_B + (\omega_0^2 + \omega_c^2)\, x_B - \omega_c^2 x_A &= 0 \end{aligned}}$$

$(\omega_c^2 = k/m)$
↳ for "coupling"

3)

Now consider adding / subtracting these two (doesn't change physics)

Add: $(\ddot{x}_A + \ddot{x}_B) + (\omega_0^2 + \cancel{\omega_c^2})(x_A + x_B) - \cancel{\omega_c^2}(x_A + x_B) = 0$ cancel

$$\boxed{(\ddot{x}_A + \ddot{x}_B) = -\omega_0^2 (x_A + x_B)}$$

$x_A + x_B$ satisfies SHO eq.! (ω/ freq $\omega = \omega_0$) 2 soln's $e^{\pm i\omega_0 t}$

Subtract: $(\ddot{x}_A - \ddot{x}_B) + (\omega_0^2 + \omega_c^2)(x_A - x_B) - \omega_c^2 (x_B - x_A) = 0$

or $(\ddot{x}_A - \ddot{x}_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) = 0$

$$\boxed{(\ddot{x}_A - \ddot{x}_B) = -(\omega_0^2 + 2\omega_c^2)(x_A - x_B)}$$

So does $x_A - x_B$! (ω/ freq $\omega = (\omega_0^2 + 2\omega_c^2)^{1/2}$) 2 soln's $e^{\pm i\sqrt{\omega_0^2 + 2\omega_c^2}\, t}$

What does this mean? Ask the demo:

1) excite so $x_A - x_B = 0 \rightarrow$ same disp. spring never compresses, so acts like 2 uncoupled oscillators (2 soln's really)

2) excite so $x_A + x_B = 0 \rightarrow$ opp. displacement spring plays role, but energy doesn't leak from one to other (ago)

we have identified 2 "Special" ways the system can oscillate "Normal Modes" these are actually all solutions $\rightarrow$ arbitrary motion from superposition

4

Consider the potential Energy:

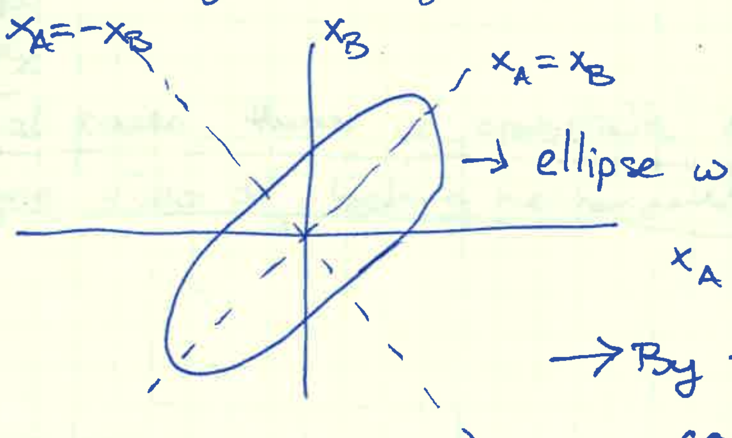
$$U = \frac{1}{2} m\omega_0^2 x_A^2 + \frac{1}{2} m\omega_0^2 x_B^2 + \frac{1}{2} k (x_A - x_B)^2$$

$$\cancel{= \frac{1}{2} m\omega_0^2 (x_A + x_B)^2 - m\omega_0^2 x_A x_B + \frac{1}{2} k (x_A^2 - 2x_A x_B + x_B^2)}$$

$$= \frac{1}{4} m\omega_0^2 \left[(x_A + x_B)^2 + (x_A - x_B)^2 \right] + \frac{1}{2} k (x_A - x_B)^2$$

$$= \underbrace{\frac{1}{4} m\omega_0^2 (x_A + x_B)^2 + \left(\frac{1}{2} k + \frac{1}{4} m\omega_0^2 \right) (x_A - x_B)^2}_{\text{equation for an ellipse!}} = U_0$$

Some chosen total max pot. energy



→ ellipse w/ major/minor axes rotated in plane

→ By taking that particular linear combination of the equations of motion, you aligned yourselves w/ major/minor axes!