AMPAD

$\boxed{\text{Lecture 5.1}}$ Quick summary: Recall damped oscillators!

$F = -kx - b\dot{x} = m\ddot{x}$ divide by m and rearrange: $\ddot{x} + \frac{b}{m}\dot{x} + \frac{kx}{m} = 0$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Solutions $Ae^{pt} \rightarrow A(p^2 \cancel{e^{pt}} + \gamma p \cancel{e^{pt}} + \omega_0^2 \cancel{e^{pt}}) = 0$

$$p^2 + \gamma p + \omega_0^2 = 0 \quad p = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$$\Rightarrow x = A_+ e^{-\gamma/2 t} e^{+i\omega t} + A_- e^{-\frac{\gamma}{2}t} e^{-i\omega t} \quad \text{with } \omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

(for $\frac{\gamma}{2} < \omega_0$) $\Rightarrow$ Decaying oscillations (underdamped)

$\frac{\gamma}{2} = \omega_0$ "Critically damped" $x = (A + Bt)e^{-\frac{\gamma}{2}t}$

↳ this sol'n is not obvious, but you can plug in and see that it works.

$\frac{\gamma}{2} > \omega_0$ "Overdamped" $x = A_+ e^{\left(-\frac{\gamma}{2} + \sqrt{(\gamma/2)^2 - \omega_0^2}\right)t} + A_- e^{\left(-\frac{\gamma}{2} - \sqrt{(\frac{\gamma}{2})^2 - \omega_0^2}\right)t}$

slow decay

rapid decay "transient"

$\boxed{\text{Damped Driven oscillators}}$

Add a driving force: linear 2nd order homogenous $\rightarrow$ linear 2nd order non-homogenous

$$F = F_0 \cos \underbrace{\omega}_{\text{frequency of driving force}} t - kx - b\dot{x} = m\ddot{x} \Rightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Complex exponentials: $x(t) \Rightarrow z(t)$ $\quad \ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}$

Solving the non-homogenous part: $z = \underset{\text{can be complex \#}}{Z_0} e^{i\omega t}$ plug in!

$$(i\omega)^2 Z_0 \cancel{e^{i\omega t}} + \gamma(i\omega) Z_0 \cancel{e^{i\omega t}} + \omega_0^2 Z_0 \cancel{e^{i\omega t}} = \frac{F_0}{m}\cancel{e^{i\omega t}}$$

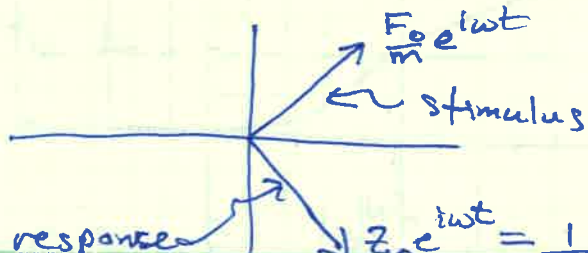
$$Z_0\left[(\omega_0^2 - \omega^2) + i\gamma\omega\right] = F_0/m \Rightarrow \boxed{Z_0 = \frac{F_0/m}{(\omega_0^2-\omega^2)+i\gamma\omega}}$$

$\boxed{\text{Let us study this graphically}}$

1) Exactly on resonance: $\omega = \omega_0 \Rightarrow Z_0 = \frac{(F_0/m)}{i\gamma\omega} = -i\frac{F_0}{m\gamma\omega}$

How to interpret this? $z = Z_0 e^{i\omega t} = \left(\frac{-i}{\gamma\omega}\right)\frac{F_0}{m} e^{i\omega t}$

$$= \left(\frac{1}{\gamma\omega}\right)\frac{F_0}{m} e^{i\omega t - i\pi/2}$$



$Z_0 e^{i\omega t} = \frac{1}{\gamma\omega}\frac{F_0}{m} e^{i\omega t - \pi/2}$ *

$\boxed{\text{Response lags stimulus by } \pi/2}$ *

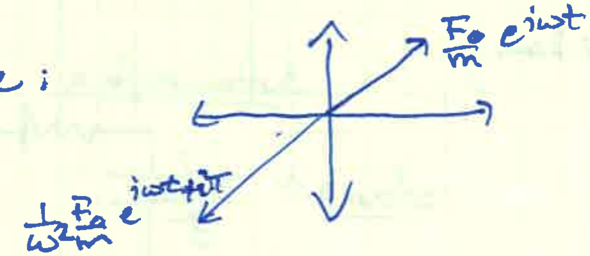
③

2) <u>Way off-resonance</u> 2 possibilities i) $\omega \gg \omega_0$

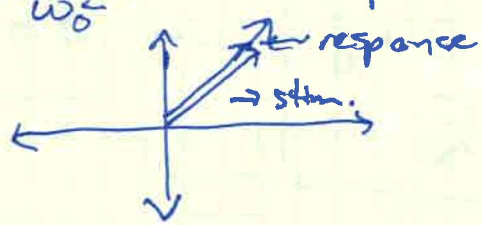
ii) $\omega \ll \omega_0$

i) $$z_0 = \frac{F_0/m}{(\omega_0^2-\omega^2)+i\gamma\omega} \xrightarrow{\omega\gg\omega_0} -\frac{F_0}{m\omega^2} \qquad z = -\frac{1}{\omega^2}\frac{F_0}{m}e^{i\omega t} = \frac{1}{\omega^2}\frac{F_0}{m}e^{i\omega t + i\pi}$$

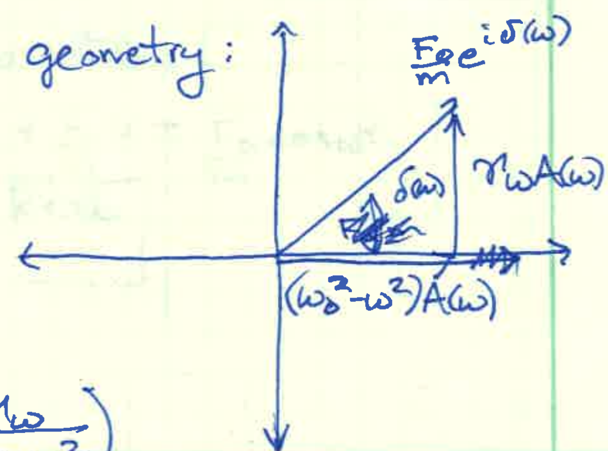
π out of phase:



ii) $$z_0 = \frac{F_0/m}{\omega_0^2-\omega^2+i\gamma\omega} \xrightarrow{\omega\ll\omega_0} \frac{F_0/m}{\omega_0^2}$$ (and assume small damping) → no phase! $$z = \frac{F_0}{m\omega_0^2}e^{i\omega t}$$



<u>Interpolate between:</u> $z_0 = A(\omega)\,e^{-i\delta(\omega)}$ geometry:



$$A(\omega) = \left[\frac{(F_0/m)^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}\right]^{1/2} \qquad \delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)$$

1)

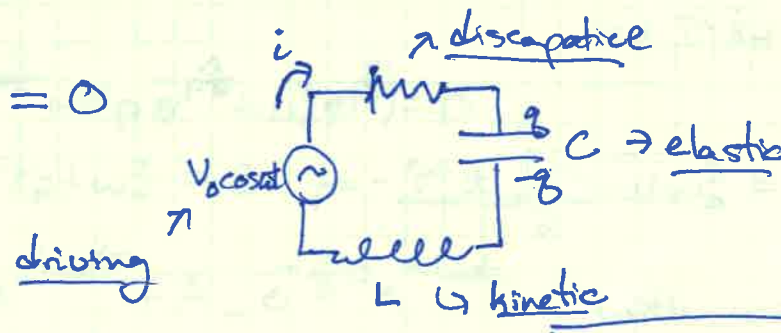
Example: Electrical Resonance

RLC circuit

$$V_C = \frac{q}{C} \qquad V_R = IR = \dot{q} R \qquad V_L = L\frac{dI}{dt} = L\ddot{q} \qquad V_{source} = V_0 \cos\omega t$$

(wall socket / function generator)

Kirchoff's law: $\sum V_i = 0$



$$V_0\cos\omega t - \dot{q}R - \frac{q}{C} - \ddot{q}L = 0$$

or

$$\boxed{\ddot{q} + \underbrace{\frac{R}{L}}_{\gamma}\dot{q} + \underbrace{\frac{1}{LC}}_{\omega_0^2} q = \frac{V_0}{L}\cos\omega t}$$

Same type of equation as mechanical forced oscillator!

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$$

$$\boxed{F_0 \Leftrightarrow V_0 \qquad m \Leftrightarrow L \qquad k \rightarrow \frac{1}{C} \qquad R \Leftrightarrow b}$$