$\boxed{\text{Lecture 12.1}}$

Outline: Superposition of Wave Pulses

Dispersion, Phase / Group Velocity

Energy / Momentum Flow in Waves

I. Superposition of Wave Pulses (Crucial for understanding wave reflection)

→ Start with slinky demonstration

Same sign pulses, then opposite sign pulses

Theory background: consider RM ~~pu~~ pulse:

$$y_1(x,t) = f(x - vt)$$

LM pulse:

$$y_2(x,t) = g(x + vt)$$

→ Both y_1 & y_2 solve wave equation

(Recall we showed any function of $x \pm vt$ solves the wave equation)

→ Wave equation is linear so superposition of LM + RM wave must also be a solution

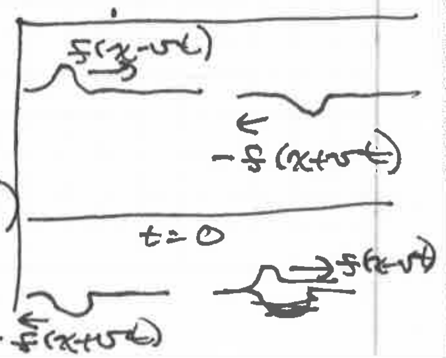
$$\boxed{y_{tot}(x,t) = y_1(x,t) + y_2(x,t) = f(x-vt) + g(v+xt) \quad \text{Solves wave equation}}$$

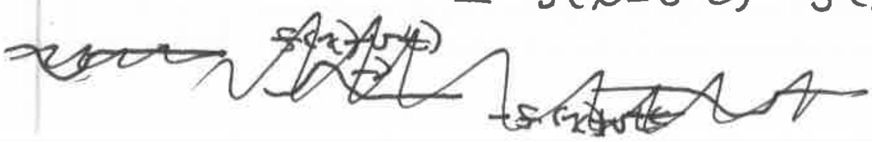
Consider $g(x) = -f(x)$ for all x

then $y(x, t=0) = f(x) + g(x) = 0$!

but at later times (or earlier)

$$y(x,t) = f(x-vt) + g(x+vt)$$
$$= f(x-vt) - f(x+vt)$$



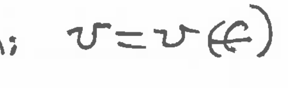


Dispersion: Phase & Group Velocities

Recall the definition of dispersion: $v = v(f)$

* wave velocity is function of frequency
  (light in materials)

Going away from ideal media → i.e. string with $v = \sqrt{\frac{T}{\mu}}$

Example the old "discrete" string 

N masses on
massless string

$A(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$, as with continuous string

but $\frac{\omega_n}{2\pi} = f_n = 2f_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$

and $v_n = \lambda_n f_n = \left(\frac{2L}{n}\right) \cdot 2f_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \neq$ constant!

* Only ~constant if $N+1 \gg n$! (Taylor expansion of sin)

$v_{high} < v_{low}$ since $\sin\left(\frac{n\pi}{2(N+1)}\right) < \frac{n\pi}{2(N+1)}$

Consequences: light being separated into colors by a prism: Snell's law: $\frac{\sin\theta_i}{\sin\theta_r} = n = \frac{c}{v}$

(light passing from vacuum to medium)

Consider: two sinusoids, Right moving, but with different wave number and velocity

$y_1 = A\sin[2\pi(k_1 x - f_1 t)]$
$y_2 = A\sin[2\pi(k_2 x - f_2 t)]$
(Same amplitude for simplicity)

$v_1 = \frac{f_1}{k_1}$; $v_2 = \frac{f_2}{k_2}$ Now superpose waves

→ Superposition gives usual beat phenomenon

$$y = y_1 + y_2 = 2A\cos\left\{\pi\left[(k_1-k_2)x - (f_1-f_2)t\right]\right\}$$
$$\times \sin\left\{2\pi\left[\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{f_1+f_2}{2}\right)t\right]\right\}$$

@ $t=0$

x

$x=0$

What happens as time advances?

Envelope (beats) have velocity $\dfrac{f_1-f_2}{k_1-k_2} = v_{group}$

Ripples w/ avg. wave number & frequency:

$$\frac{f_1+f_2}{k_1+k_2} = v_{phase}$$

Rewrite $y = 2A\cos\pi(x\,\Delta k - t\,\Delta f)\sin 2\pi(\bar{k}x - \bar{f}t)$

$$\boxed{v_{group} = \frac{\Delta f}{\Delta k} \Rightarrow \frac{df}{dk} \qquad v_{phase} = \frac{\bar{f}}{\bar{k}} \rightarrow \frac{f}{k}}$$

Group velocity is crucial → this is speed of information, energy, etc.

i.e. velocity of light in plasma: $v_{plasma} > c$!

BUT group velocity limited by c

⇒ no energy / info transport faster than c.

Example Deep water waves → dispersion is strong

⇒ Big difference between phase/group velocities

$$v_{phase} = \frac{f}{k} \Rightarrow k\,v_{phase} = f \Rightarrow dk\,v_p + k\,dv = df$$

$$\boxed{\frac{df}{dk} = v_p + k\frac{dv}{dk}}$$

Deep water: $v_p = \dfrac{c}{\sqrt{k}}$

$$v_g = \frac{d\omega}{dk} = v_p + k\frac{dv_p}{dk} = v_p - k\left(\frac{1}{2}\frac{c}{k^{3/2}}\right)$$

$$= \frac{1}{2}\frac{c}{\sqrt{k}} = \frac{1}{2}v_p$$

so $\boxed{v_g = \frac{1}{2}v_p\,!}$ Look for this next time you're looking at water (i.e. ocean) waves

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Energy and Momentum in Mechanical Waves

→ Calculate density of KE and PE along string

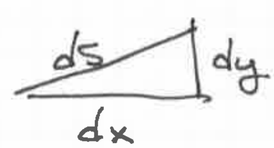
$$dKE = dE = \frac{1}{2}dm\,v^2 = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 d\ell$$

little bit of mass of short length of string ($d\ell$)

$$\boxed{\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2}$$

work done by stretching string

$$dU = -F\,d\ell = T(\underbrace{ds - dx}) = T\left(\sqrt{dx^2+dy^2} - dx\right)$$



$$\Rightarrow dU \simeq T\left(\sqrt{1+(\partial y/\partial x)^2} - 1\right)dx \simeq \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$\boxed{\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2}$$

For traveling wave $f(x \mp vt)$:

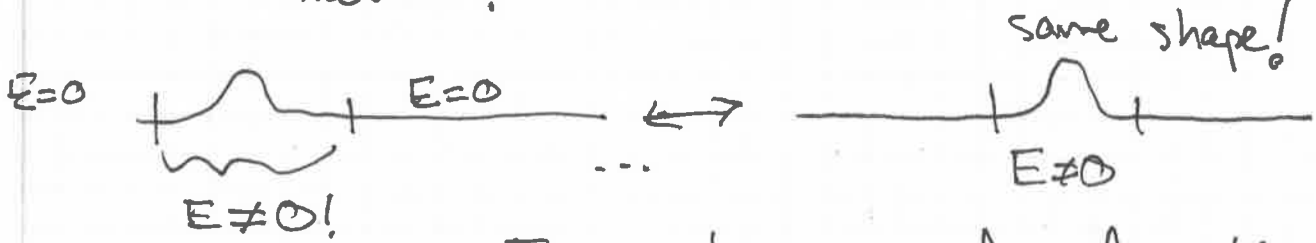
$$\frac{dK}{dx} = \frac{1}{2}\mu\left(f'(\mp v)\right)^2 = \frac{1}{2}\mu v^2\left(f'(x \mp vt)\right)^2 = \frac{1}{2}\mu\left(\frac{T}{\mu}\right)(f')^2$$

$$\frac{dU}{dx} = \frac{1}{2}T(f')^2$$

→ energy densities identical for isolated traveling wave (NOT for superposition of LM + RM, though)

→ while <u>not generally</u> true that exact equality holds instantaneously, is in keeping with <u>average</u> equipartioning of energy into KE/PE

★ Makes complete sense that energy in KE/PE is constant → shape/velocities have just moved!



Energy has <u>moved</u> along string

Waves are a mechanism of <u>energy transport</u>.
i.e. electromagnetic waves from the sun

<u>Momentum</u> will not do detailed analysis, but rather convince by demonstration