$\boxed{\text{Lecture 11.2}}$ Wave pulses

→ Waves, as we see by demonstration, need not be sinusoidal (although they can, by fourier decomposition be written as sinusoids)

"signal" can be of any arbitrary shape

In terms of fourier analysis

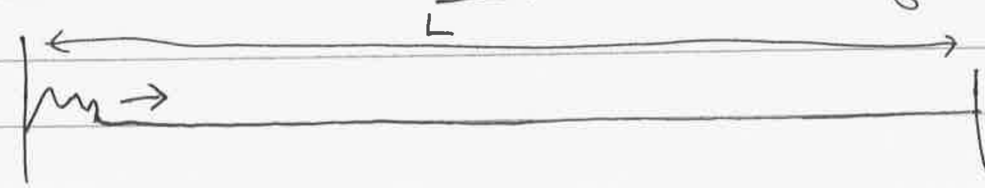
→ consider very long string with length L, mass per length μ, under tension T

Fundamental (1st) mode $T_1 = \frac{2\pi}{\omega_1} = \frac{2L}{v} = 2L\sqrt{\frac{\mu}{T}}$

> Aside note $\frac{2L}{v}$ is the time it would take a pulse moving with velocity v to make one complete circuit of the string!
>
> → makes sense given that pulse must be made out of normal modes, all w/ frequencies that are integer multiples of fundamental i.e. periodic w/ period T_1.

→ so T_1 can be made large and most excitations at one end of string will be of duration less than T_1

→ excitation is zero over most of string



②

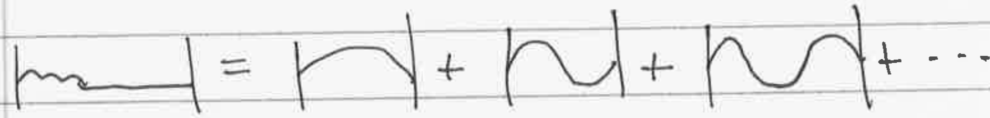
→ Spacing between harmonics: $\omega_n = \frac{n\pi v}{L}$

$$\Delta\omega = \frac{\pi v}{L} \;(= \omega_1)$$

becomes very small as L grows large

→ To make a short pulse need contribution of lots of high frequency modes

* Reason: low frequency contributions will be non-zero in flat region of string
  → need other modes to contribute negative interference in this region!



Example

What are fourier coefficients?

(zero everywhere except on rectangular pulse)

Mathematica demo $\quad B_K = \frac{2}{L}\int_0^a \left(\frac{1}{a}\right)\sin\left(\frac{n\pi x}{L}\right)$

* for small a (short pulse) B_K for large K contribute far more!

(Aside if dispersion relation is non-trivial, very short pulses ~~get~~ disperse more as they probe larger ranges of frequencies / velocities when $v = v(f)$)

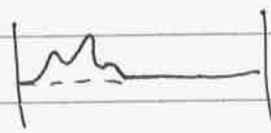
★ Bandwidth limits! ★

(3)

<u>Wave pulses with constant shape</u>

if dispersion relation is simple (v = constant for all frequencies)

then a pulse will maintain its shape as it moves

Consider pulse at time $t=0$: 

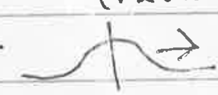
★ $y(x,t=0) = f(x)$ if pulse is moving to right ($+x$ direction) then at time t, pulse is described as $y(x,t) = f(x-vt)$

(can verify as we did for $\sin(x \mp vt)$)

left moving pulse: $y(x,t) = g(x+vt)$

(Move wave eq. discussion ★ HERE ★

<u>Example</u>: $y(x,t) = \dfrac{b^3}{b^2+(x-vt)^2}$ (rather arbitrary!)



→ nice single peak (Mathematica Demo)

<u>Question</u>: which way is the wave moving?

Transverse velocity (i.e. motion of string)

$$v_y = \frac{\partial y}{\partial t} = \frac{-b^3}{[b^2+(x-vt)^2]^2}\frac{\partial}{\partial t}\left[b^2+(x-vt)^2\right]$$

$$= \frac{2vb^3}{[b^2+(x-vt)^2]^2}$$

visualize moving pulse above!

Note: close relationship between $\frac{\partial y}{\partial t}$ and $\frac{\partial y}{\partial x}$:

$$\frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial x} \quad \text{for } +x \text{ velocity pulse}$$

$$= +v \frac{\partial y}{\partial x} \quad \text{for } -x \text{ velocity pulse}$$

Take $+x$ direction $\quad \frac{\partial y}{\partial t} = -\frac{\partial x}{\partial t}\frac{\partial y}{\partial x}$

↑ fixed value of y!
(chasing, i.e., peak of pulse)

Imagine 2 observations of string (or other medium) at different x, and different t:

$$\Delta y = \frac{\partial y}{\partial t}\Delta t + \frac{\partial y}{\partial x}\Delta x \quad \text{for small } \Delta t, \Delta x$$

$$\Rightarrow \frac{\Delta y}{\Delta t} \to \frac{dy}{dt} = \frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}$$

Gives rate of change of y as you move along at some velocity v. If $v = v_{wave}$, then

$$\Delta y = 0, \text{ and } \quad \frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$

$$\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \quad \text{"convective" derivative}$$

Wave equation consider $y(x,t) = f(x \mp vt)$

$$\frac{\partial^2 f}{\partial x^2} = f''(x \mp vt), \quad \frac{\partial^2 f}{\partial t^2} = (\mp v)^2 f''(x \mp vt)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

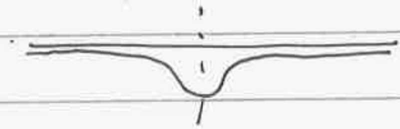
Any function of form $f(x \mp vt)$ satisfies wave equation!!

Superposition of traveling wave pulses

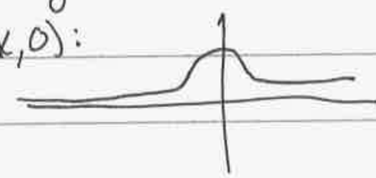
Consider two wave pulses. 1) left moving
$y_1(x,t) = g(x+vt)$
2) Right moving
$y_2(x,t) = f(x-vt)$

equal pulses, but opposite sign! $g(x) = -f(x)$

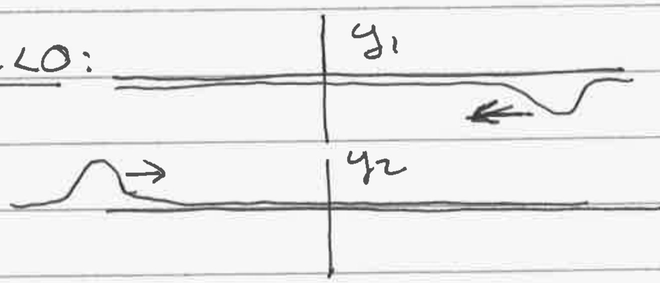
$y_1(x,0)$:



$y_2(x,0)$:



time $t<0$:



x (each moving with velocity v)

What does picture look like at time $t>0$?

A. Because of superposition principle waves just pass straight through each other!

$y = y_1 + y_2$

$t<0$:

$t=0$

$t>0$