Lecture 10.2 Fourier Analysis

→ Return to the continuous string

Considering nth normal mode:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

include phase now

Total motion = superposition

$$y(x,t) = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)\cos(\omega_n t - \delta_n)$$

phases important for general excitation!

Now consider snapshot / freeze frame: (at time $t = t_0$)

↳ string in some config at $t = t_0$

$$y(x,t_0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)\cos(\omega_n t_0 - \delta_n)$$

$$= \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \qquad \{B_n = A_n\cos(\omega_n t_0 - \delta_n)\}$$

All information here is buried in B_n's

→ any function f that obeys boundary conditions $f(0) = 0$, $f(L) = 0$ can be expressed as a sum of sines!

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{if } f(x=0,L) = 0$$

A mathematical statement (not so dissimilar to Taylor Exp.)

Lagrange and Fourier

(2)

→ Can also go the other way: Look @ particular position x_0

$$y(x_0,t) = \sum_{n=1}^{\infty} \underbrace{A_n \sin\left(\frac{n\pi x_0}{L}\right)}_{C_n} \cos(\omega_n t - \delta_n) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \delta_n)$$

Now note $\omega_n = \frac{n\pi v}{L} = n\omega_1$ ← fundamental mode frequency

$$\Rightarrow \quad T_n = \frac{2\pi}{\omega_n} = \frac{1}{n}\frac{2\pi}{\omega_1} = \frac{1}{n} T_1$$

Periodicity of $y(x_0,t)$ is T_1!

→ Any $f(t)$, periodic with period T_1, can be written like $f(t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \delta_n)$

Harmonic Analysis

So how do we calculate these coefficients?
(i.e. for Taylor expansion $f(x_0), f'(x_0), \frac{1}{2}f''(x_0), \ldots$)

Consider: $y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ → want B_n

n_1 multiply both sides by $\sin\left(\frac{n_1 \pi x}{L}\right)$ and integrate from $x=0$ to $x=L$

$$\int_0^L dx\, y(x) \sin\left(\frac{n_1\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n_1\pi x}{L}\right)$$

③

use ~~added~~ angles formula to do integral

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin\theta\sin\phi = \frac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right]$$

So $\displaystyle\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n_1\pi x}{L}\right) = \int_0^{1} dx\, \frac{1}{2}\left[\cos\left(\frac{(n-n_1)\pi x}{L}\right) - \cos\left(\frac{(n+n_1)\pi x}{L}\right)\right]$

$$= \frac{1}{2}\frac{L}{\pi}\left[\sin\left(\frac{(n-n_1)\pi x}{L}\right)\frac{1}{n-n_1} - \sin\left(\frac{(n+n_1)\pi x}{L}\right)\frac{1}{n+n_1}\right]_0^L$$

→ always integer multiples of π ! almost always ϕ, except when $n=n_1$

Then, integral is $\displaystyle\int_0^L dx \sin^2\left(\frac{n_1\pi x}{L}\right) = \int_0^L dx \left(\frac{1-\cos\frac{2n_1\pi x}{L}}{2}\right)$

$= L/2$

(also take $n_1 \approx n$ in above: $\sin\left(\frac{(n-n_1)\pi x}{L}\right)\frac{1}{n-n_1} \sim \frac{\pi x}{L}$

→ π w/ limits

⇒ $L/2$ in result)

So, get

$$\int_0^L dx\, f(x)\sin\left(\frac{n_1\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \cdot \frac{L}{2}\delta_{n,n_1} = \frac{L}{2}B_{n_1}$$

↳ 1 if $n=n_1$

0 otherwise

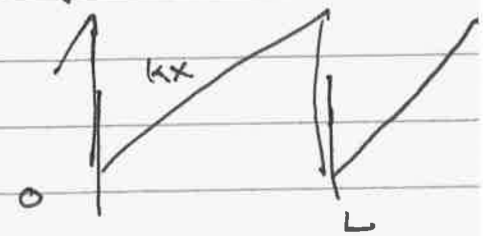
④

Master Formula for $f(x)$ with $f(x=0,L)=0$:

$$\boxed{B_n = \frac{2}{L}\int_0^L dx\, f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

→ took derivatives to get Taylor expansion coefficients ... take integrals to get "Fourier Coefficients"

Example: "Saw-tooth" $f(x) = kx$



$$B_n = \frac{2}{L}\int_0^L dx\, kx\sin\left(\frac{n\pi x}{L}\right) = \frac{2k}{\cancel{L}}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{L}{n\pi}\underbrace{\int_0^L dx\cos\left(\frac{n\pi x}{L}\right)}_{=0}\right]$$

$$= \left[-\frac{2kL}{\pi n}\cos(n\pi)\right] = -\frac{2kL}{n\pi}(-1)^n$$

So, $y(x) = kx = \frac{2kL}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \dots\right]$

Orthogonal Functions

$$\int_0^L dx\sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_2\pi x}{L}\right) = 0 \quad \text{for } n_1 \neq n_2$$
$$= \frac{L}{2} \quad n_1 = n_2$$

(like $\hat{i}\cdot\hat{j} = 0$)

Normal modes satisfy orthogonality conditions

This is ~~of~~ huge applicability in all areas of physics / engineering.