(1)

Lecture 10.1

Recall: we have been studying the motion of a stretched string fixed at both ends

L — Tension T, linear mass density μ

Wave equation:

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

$v^2 = \dfrac{T}{\mu}$ for string

Solutions: (Normal modes)

$$\boxed{\begin{gathered} y_n(x,t) = A_n \sin\left(\frac{2\pi x}{\lambda_n}\right)\cos(\omega_n t) \\ \lambda_n = \frac{2L}{n} \qquad \omega_n = \frac{n\pi v}{L} \end{gathered}}$$

* Note: Boundary conditions key in getting this form

(i.e. why no $\cos\left(\frac{2\pi x}{\lambda_n}\right)$?

Because $y(0,t) = 0$ (fixed end)

Motivates discussion of different boundary conditions

* These are key to determining the spectrum of modes! *

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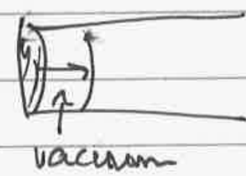
→ <u>Vibrations of air columns</u> (tube of air w/ sound waves propagating inside)

→ When an end is open, far less pressure change due to motion of air molecules (pushing into "∞" space)

Air motion there is <u>maximum</u>

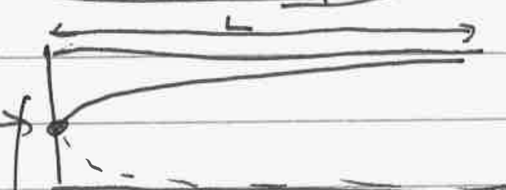
→ At closed end, air molecules cannot move (would leave vacuum behind)



Summary → Node at closed end, anti-node at open end (Plot horizontal disp. as function of x)

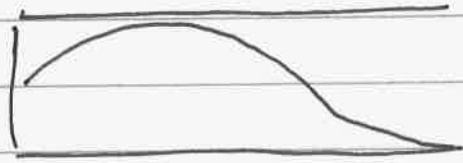
Closed / Open	Open / Open

closed

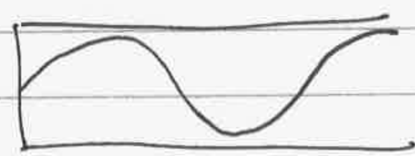


open

$\lambda = 4L$



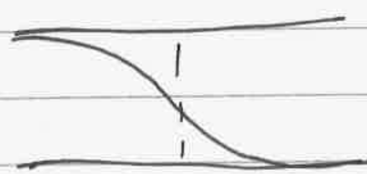
$\lambda = \frac{4}{3}L$



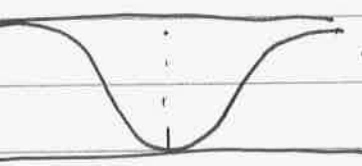
$\lambda = \frac{4L}{5}$

General

$\boxed{\lambda = \frac{4L}{(2n-1)}}$



$\lambda = 2L$



$\lambda = L$

$\lambda = \frac{2L}{3}$

General

$\boxed{\lambda = \frac{2L}{n} = \frac{4L}{(2n)}}$

③

→ We know that a wave equation governs the sound waves

$$\frac{\partial^2 \xi(x,t)}{\partial x^2} = \frac{1}{v_{sound}^2} \frac{\partial^2 \xi(x,t)}{\partial t^2}$$

(ξ is horizontal displacement of air that is at position x when at equilibrium)

For string: $v^2 = \frac{T}{\mu}$ ... What is analogous relation for sound in air or other materials?

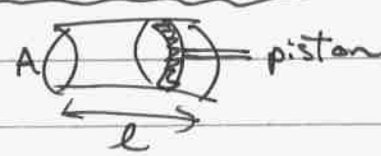
→ <u>Speed of sound</u>

Modulus of gas: $K = -V \frac{dp}{dV}$ $\left(\begin{array}{c} \Delta p = -\frac{K}{V} \Delta V \\ \text{analog of:} \\ \sim \Delta F = -k \Delta x \end{array} \right)$

↑ constant

→ <u>Note</u> temperature will change with ~~the~~ oscillation (adiabatic)

Say tube with area A, length l; has gas at pressure p, density ρ



Ideal gas: $p = \frac{1}{3} \rho v_{rms}^2$ → NOT SOUND SPEED

↳ $(\overline{v^2})$ average squared velocity

total mass $m = \rho l A \Rightarrow p = \frac{1}{3} \frac{m}{lA} v_{rms}^2$

or $p = \frac{2}{3} \frac{E_K}{lA}$ ← average KE of all gas in tube

→ Now consider moving the piston inward

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Force on piston $F_p = Ap$

Work done on gas: $\Delta W = -pA\Delta l$

distance piston is moved

Assume only place energy goes is KE of gas

$$\Delta E_k = -pA\Delta l$$

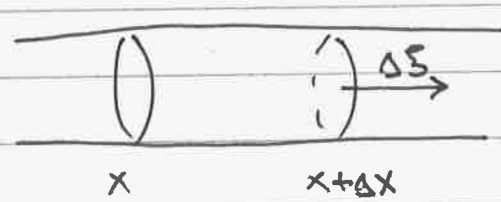
but $$\Delta p = \Delta\Big(\underbrace{\frac{2}{3A}\frac{E_k}{l}}_{p}\Big) = \frac{2}{3A}\left(\frac{\Delta E_k}{l} - E_k\frac{\Delta l}{l^2}\right)$$

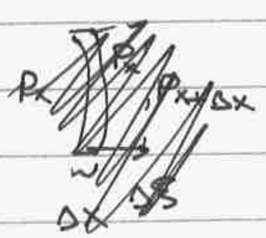
$$= \frac{2}{3A}\left(\frac{-pA\Delta l}{l} - \frac{3Alp}{2}\frac{\Delta l}{l^2}\right)$$

$$\Delta p = -\frac{5}{3}p\frac{\Delta l}{l} \rightarrow -\frac{5}{3}p\frac{\Delta V}{V}$$

rewrite: $$-V\frac{\Delta p}{\Delta V} = \frac{5}{3}p \Rightarrow \boxed{K_{adiabatic} = \frac{5}{3}p}$$

Equation of motion:





$$\Delta p_x = -\frac{K}{V}\Delta V = -\frac{K(A\Delta\xi)}{A\Delta x} = -K\frac{\Delta\xi}{\Delta x}$$

$$\rightarrow \Delta p_x = -K\frac{\partial\xi}{\partial x}$$

$$F = A\Delta p_x - A\Delta p_{x+\Delta x} = -AK\left(\frac{\partial\xi}{\partial x}\Big|_x - \frac{\partial\xi}{\partial x}\Big|_{x+\Delta x}\right)$$

$$= +AK\frac{\partial^2\xi}{\partial x^2}\Delta x$$

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→ Newton's 2nd law

$$F = AK\frac{\partial^2\xi}{\partial x^2}\Delta x = \Delta m\frac{\partial^2\xi}{\partial t^2} = \rho A\Delta x\frac{\partial^2\xi}{\partial t^2}$$

or $$\boxed{\frac{\partial^2\xi}{\partial x^2} = \frac{\rho}{K}\frac{\partial^2\xi}{\partial t^2}}$$

$$\boxed{v_{sound}^2 = \frac{K}{\rho} = \frac{\frac{5}{3}p}{\rho}} \quad \left(\approx 1.667\frac{p}{\rho}\right.$$

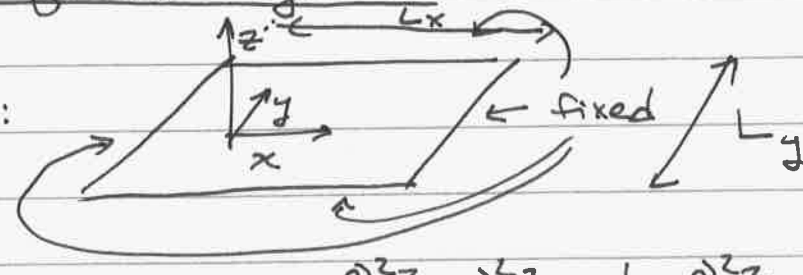
Note for many gases, relation between Δp and E_K is not so straightforward
(energy of rotation ~~of~~ particles, vibrations of molecules, etc)

generally $K = \gamma p$ for nontrivial gases

air $\gamma = 1.4$ (compare w/ 1.667...)
↑ N_2, O_2, CO_2, ...

→ Normal modes of 2D system

A square drum:



$z = z(x,y,t)$ Wave equation: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$

Solutions $z(x,y,t) = C_{n_1 n_2}\sin\left(\frac{n_1\pi x}{L_x}\right)\sin\left(\frac{n_2\pi y}{L_y}\right)\cos(\omega_{n_1 n_2}t)$

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surface tension (Force/length)

$$\omega_{n_1 n_2}^2 = \left(\frac{S}{\sigma}\right)\left[\left(\frac{n_1\pi}{L_x}\right)^2 + \left(\frac{n_2\pi}{L_y}\right)^2\right] \sim \frac{v^2 n^2 \pi^2}{L^2}$$

mass per area

$$\boxed{v^2 = \frac{S}{\sigma}}$$ $\left(\text{like } \frac{T}{\mu}\right)$ → Intuition from 1D waves on large plane

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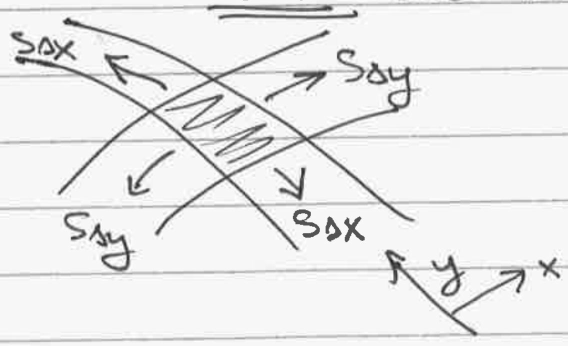
Why is wave equation $\sum_i \frac{\partial^2 \sigma}{\partial x_i^2} = \frac{1}{v^2}\frac{\partial^2 \sigma}{\partial t^2}$?

Consider path on surface:

<u>Curved</u> due to displacement



4 forces to consider

(just derive string force twice)

$$F_z = S\,\Delta x\,\Delta y\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = \sigma\,\Delta x\,\Delta y\,\frac{\partial^2 z}{\partial t^2}$$

or

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \underbrace{\left(\frac{\sigma}{S}\right)}_{1/v^2}\frac{\partial^2 z}{\partial t^2}}$$

Fourier Analysis

Back to 1D → string fixed at both ends

Consider general amp. and phase of n'th normal mode

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

Total motion = superposition of all normal modes

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

take picture (freeze frame) @ time $t = t_0$

$$y(x,t_0) = \sum_{n=1}^{\infty} \underbrace{A_n \cos(\omega_n t_0 - \delta_n)}_{\text{call this } B_n} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

string configuration at time $t = t_0$

→ Shape of string @ $t = t_0$ entirely given by coefficients B_n!