$\boxed{\text{Lecture 9.2}}$ Normal modes of Continuous Systems

→ Last time, we derived the spectrum of normal modes for N nearest-neighbor coupled identical oscillators:

Result:

Frequencies:

$$\omega_n^2 = 4\omega_0^2 \sin^2\left[\frac{n\pi}{2(N+1)}\right]$$

Amplitudes:

$$A_{p,n} = C_n \sin\left(\frac{p\,n\pi}{N+1}\right)$$

Any excitation:

→ Sum over normal modes ($n_{max} = N$)

$$y_p(t) = \sum_n A_{pn} \cos(\omega_n t + \delta_n)$$

Continuum limit: $\omega_0^2 = \dfrac{T}{\ell m} = \dfrac{T}{LM} N(N+1)$

$$p \to \frac{x}{\ell} = \frac{x}{L}(N+1)$$

$N \to \infty$

$$\boxed{\begin{aligned} \omega_n^2 &= \frac{T}{LM} n^2\pi^2 \\ A_n(x) &= C_n \sin\left(\frac{n\pi}{L}x\right) \end{aligned}}$$

Any excitation:

$$y(x,t) = \sum_{n=1}^{\infty} A_n(x) \cos(\omega_n t + \phi_n)$$

Infinite number of normal modes

②

→ Now let us work toward the same result, but starting in the continuum.

<u>Note</u> in the continuum limit, our amplitude function has intermediate zeros:

$$A_n(x) = C_n \sin\left(n\frac{\pi x}{L}\right) = 0 \quad \text{for} \quad n\pi\frac{x}{L} = \pi \underset{\text{integer}}{q}$$

or $x = L \cdot \frac{q}{n}$ for $0 \le q \le n$

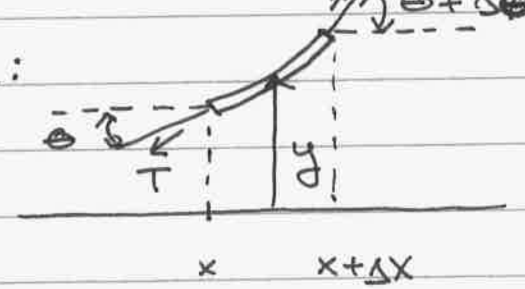
→ At such points, in a normal mode oscillation, amplitude of oscillation is <u>zero</u>

* called "<u>nodes</u>"

→ maxima of amplitude at $\frac{nx}{L}$ = half integer, are "anti-nodes"

→ Let us now study the equations of motion for a continuous string of tension T, ~~and~~ length L, and linear mass density μ.

segment of string:
(length Δx)



$$F_y = T\sin(\theta + \Delta\theta) - T\sin\theta \approx T(\theta + \Delta\theta) - T\theta = T\Delta\theta$$
$$F_x = T\cos(\theta + \Delta\theta) - T\cos\theta \approx T - T + \mathcal{O}(\theta^2) \cong 0$$

→ As always, for small displacement, <u>linear force</u>

③

Equation of motion for ~~[illegible]~~ string segment:

$$F_y = T\Delta\theta = (\mu \Delta x)\, a_y$$

now $\tan\theta = \frac{\Delta y(t)}{\Delta x} \to \frac{\partial y(x,t)}{\partial x}$ Partial derivative here is crucial!

→ take Δ of both sides:

$$\underbrace{\sec^2\theta}_{\frac{1}{\cos^2\theta}\simeq 1}\,\Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x \Rightarrow \Delta\theta \simeq \frac{\partial^2 y}{\partial x^2}\Delta x$$

Now plug back in to eq. of motion:

$$T\Delta\theta = T\frac{\partial^2 y}{\partial x^2}\Delta x = \mu\Delta x\, a_y$$

$$\to * \boxed{T\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}} *$$ Wave equation

This is precisely the second derivative we were seeing hints of last time!

note that $\frac{T}{\mu} = \frac{\partial^2 y/\partial t^2}{\partial^2 y/\partial x^2} \simeq (\text{velocity})^2$

↳ velocity of wave propagate

just like we substitute $\frac{k}{m}$ or $\frac{g}{\ell}$ or ... with ω_0^2,

we write the general wave equation as:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$ Our new favorite equation

Look for normal modes:

→ all points on ~~sting~~ string vibrating with same frequency, form $\cos(\omega t)$

Then: $y(x,t) = f(x)\cos(\omega t)$

↑ as yet undetermined function

Plug this into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$f''(x)\cancel{\cos(\omega t)} = \frac{1}{v^2} f(x)\left(-\omega^2 \cancel{\cos(\omega t)}\right)$$

$$\Rightarrow \quad f''(x) = -\frac{\omega^2}{v^2} f(x)$$

Same as SHO equation, but derivatives w.r.t. x

Solutions are sin and cos (or e^{ikx})

→ take $y=0$ (fixed) at $x=0$:

$$f(x) = A\sin\left(\frac{\omega x}{v}\right)$$

→ and we also take $y(L,t)=0$

so $\sin\left(\frac{\omega L}{v}\right) = 0$

$$\boxed{\frac{\omega_n L}{v} = n\pi}$$

↑ positive integer

Now $\omega_n^2 = \frac{v^2}{L^2} n^2\pi^2 = \frac{T}{\mu L^2} n^2\pi^2 = \frac{T}{ML} n^2\pi^2$

which agrees with our previous definition!

⑤

consider $f_n = \frac{\omega_n}{2\pi} = \frac{nv}{2L}$ the frequency (cycles per second)

now consider the wavelength (the distance over which the amplitude ~~re~~ function repeats)

$\sin\left(\frac{\omega x}{v}\right)$ repeats when $\frac{\omega x}{v} = 2\pi$ or $\frac{n\pi \cancel{v}}{L\cancel{v}}x = 2\pi$

$$\boxed{\lambda_n = \frac{2L}{n}} \quad (\text{Note } \lambda_n f_n = v)$$

↳ wavelength of n'th normal mode

Now any normal mode can be written as:

$$y_n(x,t) = A_n \sin\left(\frac{2\pi x}{\lambda_n}\right)\cos(\overset{\frac{n\pi v}{L}}{\omega_n} t + \delta_n)$$

Consider the lowest normal mode $n=1$

$\omega_1 = \frac{\pi v}{L}$ often refered to as <u>fundamental mode</u> of vibration

I.e. Violin string → E is tuned to 640 Hz ($= f_1$) and $L = .33\text{m}$, $M = 0.125\text{g}$. What is T?

$$f_1 = \frac{v}{2L} = \sqrt{\frac{T}{\mu}}\frac{1}{2L} = \sqrt{\frac{T}{(M/L)}}\frac{1}{2L} \Rightarrow T = 4f_1^2 ML$$

$$= 4(640\text{Hz})^2(.125\times 10^{-3}\text{kg})(.33\text{m})$$

$$\boxed{T \cong 68\text{N}} \quad (15\text{ lb})$$

→ Let us now peak at the case of multi-dimensional oscillation

i.e. Vibrations of a square metal plate

Can show, based on linearity of restorative forces, ~~and~~ that extension ~~to~~ beyond 1D is; for displacement σ (like y in string)

<u>2D</u> $$\boxed{\frac{\partial^2 \sigma(x,y,t)}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2}(x,y,t) = \frac{1}{v^2}\frac{\partial^2 \sigma}{\partial t^2}(x,y,t)}$$

<u>Waves on metal plate, vibrations of sheet, ...</u>

<u>3D</u> $$\boxed{\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} + \frac{\partial^2 \sigma}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \sigma}{\partial t^2}}$$

sound waves, light waves

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\sigma = \rho$ $\qquad\qquad$ $\sigma = E$ (or B)

$\downarrow$

density