Lecture 8.2

Many coupled oscillators → "infinite" degrees of freedom (a.k.a. waves in continuous media)

Analagous to $\sum \mapsto \int dx$ going from discretuum to continuum

{ Sound in air
waves on water
vibrations on string
electrical pulses in transmission lines

Vital statement: Any motion of a system of N coupled oscillators can be written as superposition of normal modes

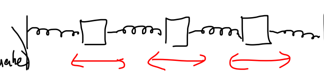
2 DoF symmetric: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \underbrace{A_+\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_+ t+\delta_+)}_{\text{Symmetric normal mode}} + \underbrace{A_-\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_- t+\delta_-)}_{\text{Anti-symm. normal mode}}$

In general: $\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} = \sum_i A_i \vec{v}_i \cos(\omega_i t+\delta_i)$

String Theory

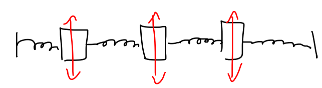
For a change, let's discuss "transverse" oscillation

Longitudinal: (sound p-waves in quake)



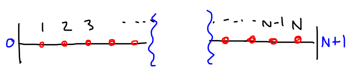
Propagation parallel with oscillation

Transverse (electromagnetic S-waves in quake)

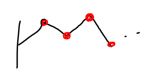


Propagation $\perp$ to oscillation

Consider massless string fixed at both ends, pulled taut, with tension T, with point masses attached

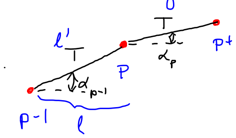


Now displace masses from equilibrium positions



tension strives to restore system to equilibrium

Zoom in and calculate! For small oscillation, can ignore increase in T due to transverse disp.



$l' = \frac{l}{\cos(\alpha)} \simeq l\left(1+\frac{\alpha^2}{2}\right)$

$\Delta T \propto \Delta l = \frac{l\alpha^2}{2}$ small

horizontal component of force on p: $-T\cos(\alpha_{p-1}) + T\cos(\alpha_p) \approx -T\left(1-\frac{\alpha_{p-1}^2}{2}\right) + T\left(1-\frac{\alpha_p^2}{2}\right) \simeq \left(\frac{\alpha_{p-1}^2-\alpha_p^2}{2}\right)T$ SMALL

vertical component: $-T\sin(\alpha_{p-1}) + T\sin(\alpha_p) \cong -T(\alpha_{p-1}-\alpha_p) = -T\left[\frac{y_p-y_{p-1}}{l} - \frac{y_{p+1}-y_p}{l}\right] = -\frac{T}{l}(2y_p - y_{p-1} - y_{p+1})$ Linear in displacement!

Newtons 2nd Law for p'th mass: $F_p = -\frac{T}{l}[2y_p - y_{p-1} - y_{p+1}] = m\ddot{y}_p$ More masses on springs!

Have N such coupled equations

$$\boxed{\ddot{y}_p + \omega_0^2[2y_p - y_{p-1} - y_{p+1}] = 0} \qquad \omega_0^2 = \frac{T}{lm}$$

2 special cases:

I. $p=1$, then $y_{p-1}=0$: $\ddot{y}_1 + \omega_0^2(2y_1 - y_2) = 0$

II. $p=N$ then $y_{N+1}=0$: $\ddot{y}_N + \omega_0^2(2y_N - y_{N-1}) = 0$

How do we get normal modes? Looks very complicated! Let us boldly forge ahead anyway

For a normal mode: $y_p = A_p\cos(\omega t)$
↑ all p have same ω!

Plug in:
$-\omega^2 A_1 + \omega_0^2[2A_1 - A_2] = 0$
$\vdots$
$-\omega^2 A_p + \omega_0^2[2A_p - A_{p-1} - A_{p+1}] = 0$
$\vdots$
$-\omega^2 A_N + \omega_0^2[2A_N - A_{N-1}] = 0$

} N equations, N+1 unknowns

One overall unconstrained amplitude

Next time: SOLUTION IS sin!