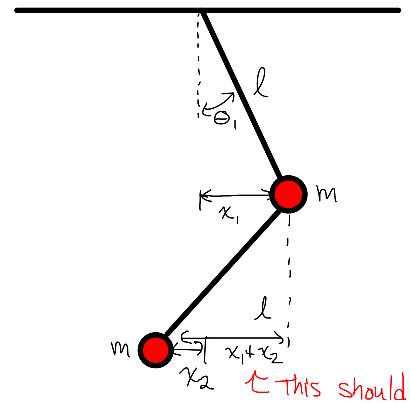
Lecture 7.2 Review of oscillating systems with 2 coupled degrees of freedom

Let us work through an example: Hallway demo → 2 sequential pendulums:



← This should have been $x_1 - x_2$ in lecture. Relative signs are crucial!!

Potential energy $U(x_1,x_2) = \underbrace{mgh_1 + mgh_2}_{\text{heights above equilibrium}}$ $\qquad h_1 \simeq \frac{x_1^2}{2\ell} \quad h_2 \simeq h_1 + \frac{(x_1 \mp x_2)^2}{2\ell}$

small angle approximation

$$U(x_1,x_2) = mg\left[\frac{x_1^2}{2\ell} + \frac{x_1^2}{2\ell} + \frac{(x_1 \mp x_2)^2}{2\ell}\right] = mg\left[\frac{x_1^2}{\ell} + \frac{(x_1 \mp x_2)^2}{2\ell}\right]$$

Equations of motion:

$$F_1 = -\frac{dU}{dx_1} = -mg\left[\frac{2x_1}{\ell} + \frac{(x_1 \mp x_2)}{\ell}\right] = -\frac{mg}{\ell}(3x_1 \mp x_2) = m\ddot{x}_1$$

$$F_2 = -\frac{dU}{dx_2} = \pm mg\left[\frac{x_1 \mp x_2}{\ell}\right] = \pm\frac{mg}{\ell}(x_1 \mp x_2) = m\ddot{x}_2$$

} COUPLED Eq. of Motion

Question Describe in as much detail as possible the normal modes of oscillation of this system

Answer Normal modes are excitations of system where each d.o.f. oscillates with same freq. ω: $x_1 = A_1\cos(\omega t)$, $x_2 = A_2\cos(\omega t)$ } Plug into coupled equations to extract info. about normal modes

☆ To characterize each normal mode, of which there are 2, find associated freq and ratio of amplitudes

Plugging in:

$$\ddot{x}_1 + \omega_0^2(3x_1 \mp x_2) = 0 \;\rightarrow\; [-A_1\omega^2 + \omega_0^2(3A_1 \mp A_2)]\cancel{\cos\omega t} = 0 \quad (1)$$

$$\ddot{x}_2 \mp \omega_0^2(x_1 \mp x_2) = 0 \;\rightarrow\; [-A_2\omega^2 \mp \omega_0^2(A_1 \mp A_2)]\cancel{\cos\omega t} = 0 \quad (2)$$

↖ g/ℓ

Using (2), have $\frac{A_1}{A_2} = -\frac{\omega^2}{\omega_0^2} \pm 1$ which we can use to simplify (1):

→ (1) Becomes $\frac{A_1}{A_2}\left(3 - \frac{\omega^2}{\omega_0^2}\right) \mp 1 = 0$ → $\left(\mp\frac{\omega^2}{\omega_0^2} \pm 1\right)\left(3 - \frac{\omega^2}{\omega_0^2}\right) \mp 1 = 0$ or $\left(\frac{\omega}{\omega_0}\right)^4 - 4\left(\frac{\omega}{\omega_0}\right)^2 + 2 = 0$ → $\boxed{\left(\frac{\omega}{\omega_0}\right)^2 = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}}$

Our 2 Normal Mode freq's!

Our 2 normal modes:

I. $\omega_+^2 = \omega_0^2(2+\sqrt{2}) = \frac{g}{\ell}(2+\sqrt{2})$ $\quad \left(\frac{A_1}{A_2}\right)_+ = -\left(\frac{\omega}{\omega_0}\right)^2 \pm 1 = -(2+\sqrt{2}) + 1 = -(1+\sqrt{2})$ $(= -2.41)$

Note Ratio is positive (masses swinging together) "Symmetric mode" analog

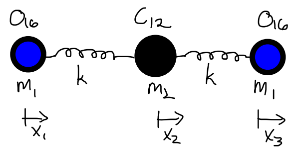
II. $\omega_-^2 = \omega_0^2(2-\sqrt{2}) = \frac{g}{\ell}(2-\sqrt{2})$ $\quad \left(\frac{A_1}{A_2}\right)_- = -\left(\frac{\omega}{\omega_0}\right)^2 \pm 1 = -(2-\sqrt{2}) + 1 = -(1-\sqrt{2})$ $(= \pm.41)$

Note Ratio is negative (masses swinging in opp. directions) "anti-symmetric" mode analog!

Remember Any motion of system can be written as superposition of the two normal modes:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_+\begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}\cos(\omega_+ t + \phi_+) + A_-\begin{pmatrix} 1-\sqrt{2} \\ -1 \end{pmatrix}\cos(\omega_- t + \phi_-)}$$

Example II CO_2 molecule



$$U = \frac{1}{2}k(x_1-x_2)^2 + \frac{1}{2}k(x_2-x_3)^2$$

$$F_1 = -\frac{dU}{dx_1} = -k(x_1-x_2) = m_1\ddot{x}_1$$

$$F_2 = -\frac{dU}{dx_2} = +k(x_1-x_2) - k(x_2-x_3) = k(x_1-2x_2+x_3) = m_2\ddot{x}_2$$

$$F_3 = -\frac{dU}{dx_3} = k(x_2-x_3) = m_1\ddot{x}_3$$

First, note $F_1+F_2+F_3=0$, so $m_1\ddot{x}_1+m_2\ddot{x}_2+m_1\ddot{x}_3=0$ No external forces on molecule!

Now use symmetry to find remaining modes: $x_1=-x_3$ gives anti-symmetric mode (1)

$x_1 = x_3$ gives symmetric mode: (2)

Study (1) first: $F_1=-F_3$ in anti-symmetric considider $F_1-F_3 = -k(x_1-x_3) = m_1\ddot{(x_1-x_3)} \Rightarrow \ddot{(x_1-x_3)} = -\frac{k}{m_1}(x_1-x_3)$

So $\boxed{x_1-x_3 = A\cos(\sqrt{k/m}\,t+\phi) \quad \text{with } x_1=-x_3}$ (Note $\ddot{x}_2=0$)

Now for (2) $F_1=F_3$ in "symmetric" mode $\rightarrow$ $F_1+F_3 = -k(x_1-2x_2+x_3) = m_1\ddot{(x_1+x_3)} = m_1\ddot{(x_1+x_3)}$

if CM is stationary, then $x_2 = -\frac{m_1}{m_2}(x_1+x_3)$ during motion

$F_1+F_3 = -k\left(1+\frac{m_1}{m_2}\right)(x_1+x_3) = m_1\ddot{(x_1+x_3)} \Rightarrow$

$$\boxed{\begin{array}{l} (x_1+x_3) = A\cos(\omega t+\phi) \quad \text{with } \omega = \sqrt{\frac{k(1+m_1/m_2)}{m_1}} = \sqrt{k\left(\frac{m_1+m_2}{m_1m_2}\right)} \\ x_2 = -\frac{m_1}{m_2}A\cos(\omega t+\phi) \qquad \left(\text{reduced mass } \mu = \frac{m_1m_2}{m_1+m_2}\right) \end{array}}$$

Now $$\boxed{\frac{\omega_+}{\omega_-} = \frac{\sqrt{k\left(\frac{m_1+m_2}{m_1m_2}\right)}}{\sqrt{k/m_1}} = \sqrt{\frac{m_1+m_2}{m_2}} = \sqrt{\frac{16+12 \text{ units}}{12 \text{ units}}} = \sqrt{\frac{7}{3}} = 1.52}$$