Lecture 6.2 Reviewing forced oscillations of system w/ 2 d.o.f.

Recall: we are driving mass A of 2 identical pendulum system with a periodic force:

$$m\ddot{x}_A + m\omega_0^2 x_A + k(x_A - x_B) = F_0 \cos\omega t$$
$$m\ddot{x}_B + m\omega_0^2 x_B + k(x_B - x_A) = 0$$

$q_1 \equiv x_A + x_B$ $\quad \ddot{q}_1 + \omega_0^2 q_1 = \frac{F_0}{m}\cos\omega t$

$\underbrace{q_2 \equiv x_A - x_B}$ $\quad \ddot{q}_2 + (\omega_0^2 + 2\omega_c^2) = \frac{F_0}{m}\cos\omega t$

In terms of new variables, equations are uncoupled $\Rightarrow$ Normal modes

SOLUTIONS (to inhomogenous part) $\quad q_1 = Q_1 \cos\omega t \quad Q_1 = \dfrac{F_0/m}{\omega_0^2 - \omega^2}$

$q_2 = Q_2 \cos\omega t \quad Q_2 = \dfrac{F_0/m}{\omega_0^2 + 2\omega_c^2 - \omega^2}$

So $x_A = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(Q_1 + Q_2)\cos\omega t$

$$\boxed{x_A = \frac{F_0}{2m}\cos\omega t\left[\frac{1}{\omega_0^2 - \omega^2} + \frac{1}{\omega_0^2 + 2\omega_c^2 - \omega^2}\right]}$$

Similarly:
$$\boxed{x_B = \frac{F_0}{2m}\cos\omega t\left[\frac{1}{\omega_0^2 - \omega^2} - \frac{1}{\omega_0^2 + 2\omega_c^2 - \omega^2}\right]}$$

Show mathematica Renditions of amplitude and phase → System has amplified response when driven at frequencies corr. to Normal Modes

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What are Normal Modes? → multiple d.o.f. moving at once w/ same freq. and phase.

1) They are special collective excitations of a system in which each dof undergoes Simple harmonic motion
2) Via superposition principle, they form a basis from which any motion of the system can be built. (like basis vectors $\hat{x}, \hat{y}, \hat{z}$, etc!)
   → fourier analysis
3) They are vibrational patterns that are excited when driving the system at certain key frequencies

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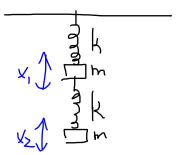
Example: 2 springs w/ masses on end, hanging from rigid ceiling → modes of oscillation? (Only up/down motion)



* Note gravitational potential energy/force is irrelevant to oscillations, setting the equilibrium position

Equations of motion: $\quad F_1 = -kx_1 - k(x_1 - x_2) = -k(2x_1 - x_2) = m\ddot{x}_1$

$F_2 = -k(x_2 - x_1) = m\ddot{x}_2$

from U: $\quad U = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - x_2)^2$

$F_1 = -\frac{dU}{dx_1} = -kx_1 - k(x_1 - x_2)$

$F_2 = -\frac{dU}{dx_2} = +k(x_1 - x_2) = -k(x_2 - x_1)$

Coupled equations $\quad \ddot{x}_1 + \omega_0^2(2x_1 - x_2) = 0$

$\ddot{x}_2 + \omega_0^2(x_2 - x_1) = 0$

Solutions will be of form:

$x_1 = A_1 \cos(\omega t)$

$x_2 = A_2 \cos(\omega t)$

Plug in $\quad \ddot{x}_1 + \omega_0^2(2x_1 - x_2) = \cos(\omega t)\left[-\omega^2 A_1 + \omega_0^2(2A_1 - A_2)\right] = 0$

$\ddot{x}_2 + \omega_0^2(x_2 - x_1) = \cos(\omega t)\left[-\omega^2 A_2 + \omega_0^2(A_2 - A_1)\right] = 0 \rightarrow \frac{A_1}{A_2} = 1 - \frac{\omega^2}{\omega_0^2}$ Plug into first equation

$-\omega^2 A_1 + \omega_0^2(2A_1 - A_2) = 0 \rightarrow \frac{(2\omega_0^2 - \omega^2)}{\omega_0^2}\frac{A_1}{A_2} - 1 = \left(2 - \frac{\omega^2}{\omega_0^2}\right)\left(1 - \frac{\omega^2}{\omega_0^2}\right) - 1 = 0$

$\left(\frac{\omega}{\omega_0}\right)^4 - 3\left(\frac{\omega}{\omega_0}\right)^2 + 1 = 0 \Rightarrow \left(\frac{\omega}{\omega_0}\right)^2 = \frac{3 \pm \sqrt{9-4}}{2} \Rightarrow \boxed{\omega^2 = \frac{k}{2m}(3 \pm \sqrt{5})}$