# HW#4 Solutions

① 4-3 in French

a) We are given a mass $m = 0.2\,\text{kg}$, hanging on a spring with $k = 80\,\text{N/m}$. There is a resistive force $F = -bv$ with $b = 4\,\frac{\text{N}\cdot\text{s}}{\text{m}}$

The equation of motion is Newton's 2nd law $F = -kx - b\dot{x} = m\ddot{x}$, or $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$, with $\gamma = \frac{b}{m} = 20\,\text{s}^{-1}$, $\omega_0^2 = \frac{k}{m} = 400\,\text{s}^{-2}$

Note that $\omega_0^2 = 400\,\text{s}^{-2} > \left(\frac{\gamma}{2}\right)^2 = 100\,\text{s}^{-2}$, which illustrates that this system is underdamped.

Solutions to this equation are $x = Ae^{pt}$, with $p^2 + \gamma p + \omega_0^2 = 0$, from above eq. of motion $\rightarrow$ $p_\pm = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$

★ Most general solution can be written as $x = A_+ e^{p_+ t} + A_- e^{p_- t}$ which can be put into form $x = A\cos(\omega t + \phi)$ by choosing $A_+ = \frac{A}{2}e^{i\phi}$, $A_- = \frac{A}{2}e^{-i\phi}$ with $\underset{\substack{\uparrow \\ \text{actual freq. of oscillation}}}{\omega} = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = \sqrt{400\,\text{s}^{-2} - 100\,\text{s}^{-2}} = 10\sqrt{3}\,\text{s}^{-1}$ So $\boxed{T = \frac{2\pi}{\omega} = \frac{\pi}{5\sqrt{3}}\,\text{s} = 0.36\,\text{s}}$

b) Under forced oscillation, the eq. of motion is augmented by the driving force: $F = -kx - b\dot{x} + F_0\cos(\omega t) = m\ddot{x}$, or $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

★ in complex exponential representation, have $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}$ → Try solution $z = Z_0 e^{i\omega t}$ as steady state (ignoring transients that damp out)

get eq. for $Z_0$: $Z_0\left[-\omega^2 + \omega_0^2 + i\omega\gamma\right] = \frac{F_0}{m}$, or $Z_0 = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\gamma\omega}$ RESPONSE at given stimulus @ freq. $\omega$ = Amplitude $A(\omega)$ + phase lag $\delta(\omega)$

$A(\omega) = |Z_0| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$ → we have $F_0 = 2\,\text{N}$, $\omega = 30\,\text{s}^{-1}$, so plug in: $\boxed{A(\omega) = \frac{(2\,\text{N})/(.2\,\text{kg})}{\left[(400\,\text{s}^{-2} - (30\,\text{s}^{-1})^2)^2 + (20 \cdot 30\,\text{s}^{-2})^2\right]^{1/2}} = 0.013\,\text{m}}$

② 4-4 In French

a) The first thing to do is convert our symbolic data into a $\gamma$ and $\omega_0^2$ (the terms in our equation of motion for damped oscillators)

Data from experiment (1) give $k$: $F = -kh + mg = 0$ → $k = \frac{mg}{h}$

equal/opp reaction of spring; applied force

Data from experiment (2) give $\gamma = \frac{b}{m}$: $F = -bu + mg = 0$ → $\gamma = \frac{b}{m} = \frac{g}{u} = \frac{g}{3\sqrt{gh}} = \frac{1}{3}\sqrt{\frac{g}{h}}$

same discussion as above; $u = 3\sqrt{gh}$

The differential equation for oscillation is $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$, or for our case, $\boxed{\ddot{x} + \frac{1}{3}\sqrt{\frac{g}{h}}\,\dot{x} + \frac{g}{h}x = 0}$ since $\omega_0^2 = \frac{k}{m} = \frac{g}{h}$ from (1)

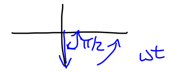
b) As in the first problem, $\omega = \sqrt{\omega_0^2 - (\gamma/2)^2}$ ★ Now we have all we need: $\omega_0^2 = \frac{k}{m} = \frac{g}{h}$, $\left(\frac{\gamma}{2}\right)^2 = \left(\frac{1}{6}\sqrt{\frac{g}{h}}\right)^2 = \frac{1}{36}\frac{g}{h}$ ⇒ $\boxed{\omega = \left[\frac{g}{h} - \frac{1}{36}\frac{g}{h}\right]^{1/2} = \sqrt{\frac{g}{h}}\sqrt{\frac{35}{36}}}$

c) The energy of the oscillator is given by the sum of KE and PE; but we can follow the max KE or PE more easily.

$PE = \frac{1}{2}kx^2 = \frac{1}{2}k\left(Ae^{-\frac{\gamma}{2}t}\cos(\omega t + \phi)\right)^2 = \frac{1}{2}kA^2 e^{-\gamma t}\cos^2(\omega t + \phi)$ ⇒ $U_{max} = \frac{1}{2}kA^2 e^{-\gamma t}$, which is down by $1/e$ when $\gamma t = 1$, or $\boxed{t = \frac{1}{\gamma} = 3\sqrt{\frac{h}{g}}}$

d) $\boxed{Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{g/h}}{\frac{1}{3}\sqrt{g/h}} = 3}$

e) The oscillator is inially at rest, so $x(0)=0$. When the bullet hits the mass, momentum will be imparted to the mass, such that $\dot{x}>0$ @ $t=0$. Thus, $\delta=\pi/2$:



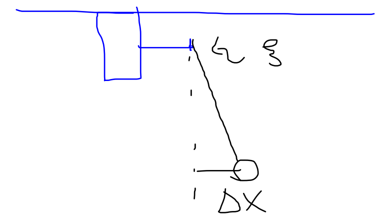

Real part starts at zero, becomes positive.

f) Response is $\boxed{A(\omega)=\frac{F_0/m}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{1/2}}=\frac{g}{[(g/h-\frac{2g}{h})^2+(\frac{1}{3}\sqrt{\frac{g}{h}}\cdot\sqrt{\frac{2g}{h}})^2]^{1/2}}=\frac{g}{[(\frac{g}{h})^2+\frac{2}{9}(\frac{g}{h})^2]^{1/2}}=\frac{3h}{\sqrt{11}}}$ (As we derived in first problem)

③ 4-5 in French

a) By getting this data, we estimate $Q=\pi n=\pi(50)=157$ and so $\omega_0\approx\omega$ (true for large $Q$)

now $Q=\frac{\omega_0}{\gamma}=\frac{\sqrt{g/\ell}}{\gamma}=157$ so $\gamma=\frac{1}{157}\sqrt{\frac{g}{\ell}}$. Now we want to obtain the equations of motion:




Without driving, the equation of motion is $\ddot{x}+\gamma\dot{x}+\omega_0^2x=0$.

$F=-m\omega_0^2(x-\xi)-b\dot{x}=m\ddot{x}$

or $\ddot{x}+\gamma\dot{x}+\omega_0^2x=\omega_0^2\xi$

$$\boxed{\ddot{x}+\gamma\dot{x}+\frac{g}{\ell}x=\frac{g}{\ell}\xi}$$

$F=-m\omega_0^2\Delta x-b\dot{x}=m\ddot{x}$

actual vel. + accel

$\Delta x=x-\xi$

b) at resonance $\omega=\omega_0$ response amplitude $A(\omega)=\frac{g/\ell\,\xi_0}{\gamma\omega_0}=\frac{\omega_0}{\gamma}\xi_0=Q\xi_0=157\,\xi_0=157\text{mm}$

c) The general response amplitude is $A=\frac{\omega_0^2\xi_0}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{1/2}}$ to get $A=\frac{1}{2}A_{max}=\frac{\omega_0\xi_0}{\gamma}$, need $(\omega_0^2-\omega^2)^2+(\gamma\omega)^2=4(\gamma\omega)^2$, or $(\omega^2-\omega_0^2)^2=3(\gamma\omega)^2$

or $\left(\frac{\omega}{\omega_0}\right)^4-2\left(\frac{\omega}{\omega_0}\right)^2+1=\frac{3}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2$

Solving: $\boxed{\left(\frac{\omega}{\omega_0}\right)^2=\frac{2+\frac{3}{Q^2}\pm\sqrt{(2+3/Q^2)^2-4}}{2}\simeq 1\pm\frac{\sqrt{3}}{Q}}$ where I have assumed large $Q$

that is, $\boxed{\omega^2=\frac{g}{\ell}\cdot\left(1\pm\frac{\sqrt{3}}{157}\right)}$ @ half-max response

④ 4-7 in French

The solution we study is $x=A\cos(\omega t-\delta)$

a) $KE=\frac{1}{2}m\dot{x}^2=\frac{1}{2}mA^2\omega^2\sin^2(\omega t-\delta)$

b) $PE=\frac{1}{2}kx^2=\frac{1}{2}kA^2\cos^2(\omega t-\delta)$

c) note $\langle\sin^2\omega t\rangle=\langle\cos^2(\omega t)\rangle=\frac{1}{2}$ from $\langle x(t)\rangle=\frac{1}{T}\int_0^T x(t)\,dt$

Then $\langle KE\rangle=\frac{1}{4}mA^2\omega^2$ and $\langle PE\rangle=\frac{1}{4}kA^2$ so $\boxed{\frac{\langle KE\rangle}{\langle PE\rangle}=\frac{m\omega^2}{k}=\frac{\omega^2}{\omega_0^2}}$

d) The $\langle KE\rangle+\langle PE\rangle$ are equal on resonance! $\omega=\omega_0$. The instantaneous $E$ is then $E=KE+PE=\frac{1}{2}m\omega_0^2A^2\sin^2(\omega t+\phi)+\frac{1}{2}kA^2\cos^2(\omega t+\phi)$

$=\frac{1}{2}kA^2$ (constant)

e) $E_{tot}=\frac{1}{2}m\omega^2A^2\sin^2(\omega t-\delta)+\frac{1}{2}m\omega_0^2\cos^2(\omega t-\delta)$ $\quad\dot{E}_{tot}=m\omega^3A^2\sin(\omega t-\delta)\cos(\omega t-\delta)-m\omega_0^2\omega\cos(\omega t-\delta)(\sin\omega t-\delta)=0$ only if $\omega(\omega^2-\omega_0^2)=0\ldots$ ie $\omega=0,\omega_0$

$=\frac{1}{2}m\omega_0^2A^2+\frac{1}{2}m(\omega^2-\omega_0^2)A^2\sin^2(\omega t-\delta)$

⑤ 4-13 in French

a) You are given a resonance curve in terms of energy input. Now Power $\sim \text{Amp}^2 \sim \dfrac{(F_0/m)^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}$ @ ½ max when $(\omega_0^2-\omega^2)^2=(\gamma\omega)^2$; or $\left(\frac{\omega}{\omega_0}\right)^4-\left(2+\frac{1}{Q^2}\right)\left(\frac{\omega}{\omega_0}\right)^2+1=0$

$$\left(\frac{\omega}{\omega_0}\right)^2 \simeq 1+\frac{1}{2Q^2} \pm \sqrt{\frac{1}{4}\left(2+1/Q^2\right)^2-1}$$
$$\simeq 1 \pm \frac{1}{2Q} \Rightarrow \omega^2=\omega_0^2\left(1\pm 1/2Q\right)$$
$$\Rightarrow \Delta\omega=\omega_0 1/2Q$$

Clearly, max is at $\omega=40\,s^{-1} \Rightarrow \boxed{\omega_0=40\,s^{-1}}$ and $\Delta\omega = 1\,s^{-1} = \dfrac{40\,s^{-1}}{2Q} \Rightarrow \boxed{Q=20}$

b) Remember, for $1/e$, $n=\dfrac{Q}{\pi}$ so for $\dfrac{1}{e^5}$, $\boxed{n=\dfrac{5Q}{\pi}=\dfrac{100}{\pi}=31.8}$