$\boxed{\text{Lecture 6.1}}$

Last time   system with 2 coupled degrees of freedom: 2 pendula, spring connecting them

We know, close to equilibrium, forces are linear in displacement $F_A = -m\omega_0^2 x_A - k(x_A - x_B) = m\ddot{x}_A$

$F_B = -m\omega_0^2 x_B - k(x_B - x_A) = m\ddot{x}_B$

Coupling $\Leftarrow$ x_A and x_B equations are interdependant   Coupling $\Rightarrow$ motion of one influences the other!

Universal behavior   Near equilibrium, equations are always linear   (potential function always quadratic $\rightarrow$ $F_i = -\frac{dU}{dx_i}$ ELLIPSOIDAL ISOPOTENTIAL CURVES)

For pendulums + sping: $U(x_A, x_B) = \frac{1}{2} m\omega_0^2 x_A^2 + \frac{1}{2} m\omega_0^2 x_B^2 + \frac{1}{2} k (x_A - x_B)^2$ Off-kilter ellipse major/minor axes not horizontal/verticle

Consider summing/subtracting equations of motion:

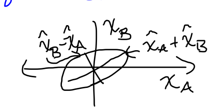
1) Add: $(\ddot{x_A + x_B}) + \omega_0^2 (x_A + x_B) = 0$ $x_A + x_B = X_+ \cos(\omega_0 t + \alpha_+)$: Symmetric "normal mode"

2) Subtract: $(\ddot{x_A - x_B}) + (\omega_0^2 + 2\underbrace{\omega_c^2}_{k/m})(x_A - x_B) = 0$ $x_A - x_B = X_- \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \alpha_-)$: Anti-symmetric

Any motion is linear superposition of these two solutions (just like two independant SHO's!)

Coupled degrees of freedom   SUMMARY

1) near equilibrium, all systems (even N d.o.f.) are generally coupled SHO's
2) equations are linear, but mixed
3) Can always find combinations of equations where equations are un-coupled   FIND THE NORMAL MODES $\rightarrow$ Same # as d.o.f. (various resonant frequencies) + special motions
4) equations uncoupled along major/minor, etc. axes of ellipsoids of iso-potential energy curves



Superposing Normal Modes   Let us work out solution given more general initial conditions

Consider: @ $t=0$: $x_A = x_A^0$, $x_B = 0$

$\dot{x}_A = 0$, $\dot{x}_B = 0$

General Sol'n

$x_A + x_B = A_+ \cos(\underbrace{\omega_+}_{\omega_0} t + \delta_+) = X_+(t)$

$x_A - x_B = A_- \cos(\omega_- t + \delta_-) = X_-(t)$

Apply initial conditions

$\dot{x}_A(t=0) = -\frac{1}{2}(\omega_+ A_+ \sin(\delta_+) + \omega_- A_- \sin(\delta_-))$

$\dot{x}_B(t=0) = -\frac{1}{2}(\omega_+ A_+ \sin(\delta_+) - \omega_- A_- \sin(\delta_-))$ $\Big\}$ $\delta_+ = \delta_- = 0$

$x_A(t=0) = \frac{1}{2}(X_+(0) + X_-(0)) = \frac{1}{2}(A_+ \cos(\delta_+) + A_- \cos(\delta_-)) = \frac{1}{2}(A_+ + A_-) = x_A^0$

$x_B(t=0) = \frac{1}{2}(X_+(0) - X_-(0)) = \frac{1}{2}(A_+ \cos(\delta_+) - A_- \cos(\delta_-)) = \frac{1}{2}(A_+ - A_-) = 0$

$A_+ = A_- = x_A^0$

Final Solution $x_A = \frac{x_A^0}{2}[\cos(\omega_+ t) + \cos(\omega_- t)]$ $x_B = \frac{x_A^0}{2}[\cos(\omega_+ t) - \cos(\omega_- t)]$

BEATS: $x_A = x_A^0 \cos(\bar{\omega} t) \cos(\frac{1}{2} \Delta\omega t)$ $(\omega_- - \omega_+)$ $x_B = x_A^0 \cos(\bar{\omega} t - \pi/2) \sin(\frac{1}{2} \Delta\omega t)$

Forced Coupled Oscillators

Consider Driving oscillator A w/ sinusoidally varying force F_0

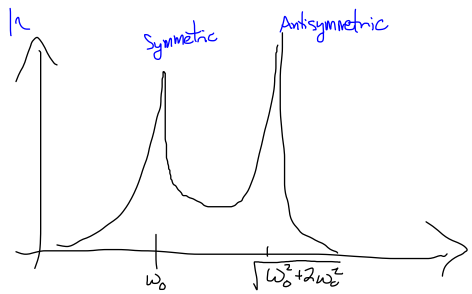
$$m\ddot{x}_A + m\omega_0^2 x_A + k(x_A - x_B) = F_0 \cos \omega t \qquad q_1 = x_A + x_B: \quad \ddot{q}_1 + \omega_0^2 q_1 = \frac{F_0}{m}\cos(\omega t)$$

$$m\ddot{x}_B + m\omega_0^2 x_B + k(x_B - x_A) = 0 \qquad q_2 = x_A - x_B: \quad \ddot{q}_2 + (\omega_0^2 + 2\omega_c^2) q_2 = \frac{F_0}{m}\cos(\omega t)$$

Solutions are relatively simple $\rightarrow$ $q_1 = Q_1 \cos \omega t \rightarrow Q_1(-\omega^2 + \omega_0^2) = F_0/m \Rightarrow Q_1 = \frac{F_0/m}{(\omega_0^2 - \omega^2)}$ $\quad x_A = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(Q_1 + Q_2)\cos \omega t$

$q_2 = Q_2 \cos \omega t \rightarrow Q_2(-\omega^2 + \omega_0^2 + 2\omega_c^2) = \frac{F_0}{m} \quad Q_2 = \frac{F_0/m}{(\omega_0^2 + 2\omega_c^2 - \omega^2)}$ $\quad x_B = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}(Q_1 - Q_2)\cos(\omega t)$

Show Plots of x_A & x_B amplitude, phase:



Introducing damping is not as big a deal as book implies:

$$\ddot{q}_1 + \gamma \dot{q}_1 + \omega_0^2 q_1 = \frac{F_0}{m} e^{i\omega t} \qquad Q_1 = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

$$\ddot{q}_2 + \gamma \dot{q}_2 + (\omega_0^2 + 2\omega_c^2) q_2 = F_0/m \, e^{i\omega t} \qquad Q_2 = \frac{F_0/m}{(\omega_0^2 + 2\omega_c^2 - \omega^2) + i\gamma m}$$

Algebra nasty, but:

$\frac{1}{2}(Q_1 \pm Q_2)$ gives complex amplitudes

(mag. & phase of x_A & x_B)