$\boxed{\text{HW\#3 Solutions}}$

1) $I=\sum_i m_i \vec{r}_i^{\,2} \rightarrow \int dm\, \vec{r}^{\,2} = \int_{-\ell/3}^{2/3\,\ell} \lambda\, dx\, x^2 = \lambda \frac{x^3}{3}\Big|_{-\ell/3}^{\frac{2\ell}{3}} = \frac{\lambda \ell^3}{9} = \frac{M\ell^2}{9}$ or you can use parallel axis thm: $I = I_{cm} + M\underbrace{d^2}_{\text{dist to C.M.}}$ (book uses this)

$$= \int_{-\ell/2}^{\ell/2} \lambda x^2 dx + Md^2 = \frac{M\ell^2}{12} + M\left(\frac{\ell}{6}\right)^2 = \frac{M\ell^2}{9}$$

Energy eq: $E=\frac{1}{2}I\dot{\theta}^2+\frac{1}{2}Mgd\,\theta^2 = \frac{1}{2}\frac{M\ell^2}{9}\dot{\theta}^2+\frac{1}{2}Mg\frac{\ell}{6}\theta^2$ Energy conservation: $0=\dot{E}=\frac{M\ell^2}{9}\ddot{\theta}\dot{\theta} + Mg\frac{\ell}{6}\dot{\theta}\theta = 0$

$$\frac{M\ell^2}{9}\ddot{\theta} + \frac{Mg\ell}{6}\theta = 0 \qquad \ddot{\theta} = -\underbrace{\frac{3}{2}\frac{g}{\ell}}_{\omega_0^2}\theta$$

$\omega_0 = \sqrt{\frac{3g}{2\ell}}$, $\boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{2\ell}{3g}}}$

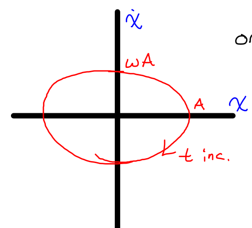
2) Use ∥ axis thm to get $I = I_{cm} + Mr^2 = Mr^2 + Mr^2 = 2Mr^2 = 2M\left(\frac{d}{2}\right)^2 = \frac{Md^2}{2}$ Then $E=\frac{1}{2}I\dot{\theta}^2 + Mg\frac{d}{2}\theta^2 = \frac{1}{2}\frac{Md^2}{2}\dot{\theta}^2+\frac{1}{2}Mgd\,\theta^2$

or $\ddot{\theta} = -\frac{2g}{d}\theta \Rightarrow \omega_0 = \sqrt{\frac{2g}{d}}$ $\boxed{T = 2\pi\sqrt{\frac{d}{2g}}}$

3) a) for an undamped oscillator, $x = A\cos(\omega t+\alpha)$, $\dot{x} = -A\omega\sin(\omega t+\alpha)$ note that $x^2+\left(\frac{\dot{x}}{\omega}\right)^2 = A^2\cos^2(\omega t+\alpha)+\frac{(A\omega)^2}{\omega^2}\sin^2(\omega t+\alpha) = A^2$

or $\frac{x^2}{A^2}+\frac{\dot{x}^2}{(A\omega)^2} = 1$

This is ellipse w/ major/minor axes w/ length A, and ωA:



4) Verify $x(t) = Ae^{-\alpha t}\cos\omega t$ is sol'n of damped SHO eq. $\ddot{x}+\gamma\dot{x}+\omega_0^2 x=0$ → consider complex solutions $z = Ae^{pt}$

$z(p^2+\gamma p+\omega_0^2)=0$, or $p = -\frac{\gamma}{2}\pm\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}$, or $p=-\frac{\gamma}{2}\pm i\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}$ General sol'n, $z = A_+e^{p_+t}+A_-e^{p_-t}$

$$= A_+e^{-\frac{\gamma}{2}t}e^{i\omega t} + A_-e^{-\frac{\gamma}{2}t}e^{-i\omega t}$$

$$\omega = \sqrt{\omega_0^2-(\gamma/2)^2}$$

Now take $A_+ = A_- = \frac{A}{2}$, then $z(t) = \frac{A}{2}e^{-\frac{\gamma}{2}t}\left[e^{i\omega t}+e^{-i\omega t}\right] = Ae^{-\gamma/2\,t}\cos(\omega t)$

$\boxed{\text{This has form } Ae^{-\alpha t}\cos(\omega t) \text{ with } \alpha = \gamma/2,\ \omega=\sqrt{\omega_0^2-(\gamma/2)^2}}$

5) The damped SHO eq is $\ddot{x}+\gamma\dot{x}+\omega_0^2x=0$ now call units of x $[x]=d$ (could be position, volts, etc)

each term, same units

Then: $[\dot{x}] = \frac{d}{s}$, $[\ddot{x}] = \frac{d}{s^2}$ To match, $[\gamma] = \frac{1}{s}$, so $[\gamma/\omega_0] = 1$

(NO UNITS)

For an underdamped oscillator, amplitude decays as $e^{-\gamma/2\,t} = e^{-\frac{\omega_0 t}{2Q}}$. Now write $t = n\overset{\text{\#cycles}}{}\underset{}{}T\overset{\text{period}}{} = \frac{2\pi n}{\omega}$ So $e^{-\gamma/2\,t}\sim e^{-\frac{n\pi}{Q}}$

So Amp goes down by 1/e when $n \simeq \frac{Q}{\pi}$ so $\boxed{Q \sim \pi\times(\text{\# of cycles for Amp. to decay by } 1/e)}$